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A Uniform Framework for Modeling Nondeterministic, Probabilistic, Stochastic, or Mixed Processes and their Behavioral Equivalences

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Abstract

Labeled transition systems are typically used as behavioral models of concurrent processes. Their labeled transitions define a one-step state-to-state reachability relation. This model can be generalized by modifying the transition relation to associate a state reachability distribution with any pair consisting of a source state and a transition label. The state reachability distribution is a function mapping each possible target state to a value that expresses the degree of one-step reachability of that state. Values are taken from a preordered set equipped with a minimum that denotes unreachability. By selecting suitable preordered sets, the resulting model, called ULTraS from Uniform Labeled TRAnsition System, can be specialized to capture well-known models of fully nondeterministic processes (LTS), fully probabilistic processes (ADTMC), fully stochastic processes (ACTMC), and nondeterministic and probabilistic (MDP) or nondeterministic and stochastic (CTMDP) processes. This uniform treatment of different behavioral models extends to behavioral equivalences. They can be defined on ULTraS by relying on appropriate measure functions that express the degree of reachability of a set of states when performing multi-step computations. It is shown that the specializations of bisimulation, trace, and testing equivalences for the different classes of ULTraS coincide with the behavioral equivalences defined in the literature over traditional models except when nondeterminism and probability/stochasticity coexist; then new equivalences pop up.

Keywords: labeled transition systems, discrete-/continuous-time Markov chains, discrete-/continuous-time Markov decision processes, bisimulation equivalences, trace equivalences, testing equivalences.

1. Introduction

Process algebras (see [15] and references therein) have been successfully used in the last thirty years to model and analyze the behavior of concurrent systems. Apart from specific syntactic operators used to define the term algebra, the basic ingredients of these formalisms are the model called *labeled transition system (LTS)* [81] and *behavioral relations* in the form of equivalences or preorders. By exploiting the structural operational semantic approach [104], an LTS is compositionally associated with each term and behavioral relations over LTS models are introduced to compare process terms describing systems at different levels of abstraction and to investigate properties of interest.

Initially, process algebras were mainly designed to model and assess functional behaviors. However, it was soon noticed that other aspects of concurrent systems are at least as important as the functional ones. Thus, many variants of process algebras have been introduced to take into account quantitative aspects of concurrent systems. There have been proposals of *(deterministically) timed* process algebras, *probabilistic* process algebras, and *stochastic(ally timed)* process algebras, whose semantics have been rendered in terms of richer LTS models quotiented with appropriate behavioral relations.

The purpose of this paper is to set up a unifying semantic framework based on an extension of the LTS model and to provide a uniform definition of the main behavioral relations, which underpins the many nondeterministic, probabilistic, timed, and mixed variants of process algebras that have appeared in the literature. Our work builds on two existing extended models – rate transition systems and simple probabilistic automata – which we now briefly recall.

In [48], two of the authors of the present paper, together with Latella and Massink, introduced a variant of the LTS model called *rate transition system (RTS)* as a uniform tool for providing semantics to stochastic process languages. The transition relation of LTS describes the evolution of a system as a set of triples (s, a, s') each expressing reachability in one step of a state s' from a state s when executing action a. The transition relation $\rightarrowtail$ of RTS, instead, associates with any pair, consisting of a source state s and an action a, a function $\mathcal{P}$ mapping each possible target state s' into a nonnegative real number. The RTS transition $s \overset{a}{\rightarrowtail} \mathcal{P}$ has the following meaning: if $\mathcal{P}(s') = v > 0$, then s' is reachable from s by executing a at rate v; if $\mathcal{P}(s') = 0$, then s' is not reachable from s via a. In [49], the same authors provided an elegant RTS-based operational semantics for TIPP [63], PEPA [73], EMPA [27], and IML [69] (as representatives of stochastic languages with a multi-way interaction paradigm) and for stochastic CCS and stochastic π-calculus [105] (as examples of stochastic languages based on the two-way interaction paradigm).

In [109], Segala introduced a variant of Rabin's probabilistic automata [107] whose transition relation associates a discrete probability distribution over actions and target states with any source state. Every transition is thus of the form $(s, \mathcal{P})$ where s is the source state and $\mathcal{P}$ associates a probability value with each pair (a, s') formed by an action a and a target state s'. This results in a model combining probability and nondeterminism, which is fully probabilistic when every state has at most one outgoing transition and fully nondeterministic when every transition leads to a probability distribution concentrating all the probability mass on a single pair. A special case of this model is the so-called *simple probabilistic automaton*, in which every transition leads to a probability distribution concentrating all the probability mass on pairs with the same action a and hence can be expressed as $s \xrightarrow{a} \mathcal{P}$ where $\mathcal{P}$ is a discrete probability distribution over target states only. For simple probabilistic automata, notions of bisimulation, trace, and testing equivalence were studied by Segala and coauthors in [112, 110, 111].

In this paper, we perform a further step in the direction of a uniform characterization of the semantics of different process calculi by developing a generalization of RTS models and simple probabilistic automata that uses the same format as their transition relations, i.e., the third element of each transition is a function over states and not just a single state. The model we propose is called ULTRAS from U*niform* L*abeled* TRA*nsition* S*ystems*. Its transition relation associates with any pair of source state and transition label (s, a) a function $\mathcal{D}$ mapping each possible target state into an element of the support D of a preordered set equipped with a minimum denoted by $\perp_D$. Given a transition $s \xrightarrow{a} \mathcal{D}$, the value of $\mathcal{D}(s')$ expresses the degree of *one-step reachability* of s' from s via that a-transition; if $\mathcal{D}(s') = \perp_D$ then s' is not reachable from s via that a-transition.

The ULTRAS model can be used to capture different classes of processes by appropriately choosing D. In particular, we will see that we capture:

1. Fully nondeterministic processes, if D is the support set $\mathbb{B} = \{\perp, \top\}$ of the traditional Boolean algebra.
2. Fully probabilistic processes and processes combining nondeterminism and probability, if $D = \mathbb{R}_{[0,1]}$.
3. Fully stochastic processes and processes combining nondeterminism and stochasticity, if $D = \mathbb{R}_{\geq 0}$.

As stressed at the beginning of this section, modeling state transitions and their annotations is only one of the key ingredients of the description of concurrent processes. One must also combine single transitions into computations and find out ways for determining when two states give rise to behaviorally equivalent computation trees. In this paper, we focus on the three major approaches to the development of behavioral equivalences and we define bisimulation, trace, and testing equivalences for the ULTRAS model. We restrict, for the moment, attention to their strong version, i.e., we assume that all actions are observable.

An important component of our definitions of the three equivalences is a *measure function* $\mathcal{M}_M(s, \alpha, S')$ that returns elements of the support M of another preordered set equipped with a minimum. This function computes the degree of *multi-step reachability* of a set of target states S' from a source state s when performing computations labeled with the sequence of actions α. We will see that, to capture classical equivalences for the different classes of processes, we need different measure functions:

1. For nondeterministic processes, the measure of a computation from s to S' labeled with α is $\top$ if the computation exists and $\bot$ otherwise.
2. For probabilistic processes, the measure function yields a value in $\mathbb{R}_{[0,1]}$ that represents the probability of the set of computations from s to S' labeled with α.
3. For stochastic processes, to capture the different equivalences proposed in the literature, we have to distinguish two cases:
   - In the *end-to-end* case, given a time threshold $t \in \mathbb{R}_{\geq 0}$, the measure function yields a value in $\mathbb{R}_{[0,1]}$ that represents the probability that the set of computations labeled with α leads from s to S' within t time units.
   - In the *step-by-step* case, given a sequence of time thresholds $t_i \in \mathbb{R}_{\geq 0}$, the measure function yields a value in $\mathbb{R}_{[0,1]}$ that represents the probability that the set of computations labeled with α leads from s to S' within t_i time units for each step i.

One of the main objectives of this work is assessing the different choices that have been presented in the literature in the last twenty years for generalizing behavioral equivalences over LTS models to richer models. We consider it interesting to see which of them are naturally captured by our general approach and which ones need, instead, an ad hoc treatment. We will see that, as long as we confine ourselves to considering models that deal with purely nondeterministic, purely probabilistic, or purely stochastic processes, the known (and generally accepted) equivalences are directly captured. For models that combine probability or stochasticity with nondeterminism, the situation is less straightforward. There are many different ways of interpreting such combinations that influence the way behavioral equivalences are defined, leading to an explosion of potential approaches.

Interestingly enough, for mixed processes we will see that our abstract approach leads to new equivalences that were not known in the literature. More precisely, our variant of probabilistic bisimulation equivalence, which has been studied in [26], has a strong connection with PML, the simple probabilistic extension of Hennessy-Milner logic that is in agreement with probabilistic bisimilarity for probabilistic processes without internal nondeterminism [85]. In contrast, the probabilistic bisimulation equivalence in [112] corresponds to a much richer modal logic with a specific operator for capturing probability measures of states reachability [70]. Moreover, our variant of probabilistic testing equivalence, which has been studied in [25], is a conservative extension of the nondeterministic testing equivalence in [47] also when both nondeterministic and probabilistic tests are used, and implies probabilistic trace equivalence also under deterministic schedulers. This is not the case with the probabilistic testing equivalences in [129, 77, 111, 52]. Finally, our variant of probabilistic trace equivalence, which has been studied in the full version of [25], is a congruence with respect to parallel composition, while the probabilistic trace equivalence in [110] is not compositional.

The rest of the paper, which extends [23] in that it considers also mixed processes and testing equivalences, is organized as follows. In Sect. 2, we define the ULTraS model and show that suitable specializations of this model coincide with seven models that are widely used to describe fully nondeterministic processes, fully probabilistic processes, fully stochastic processes, and processes combining nondeterminism and probability or nondeterminism and stochasticity. In Sect. 3, we provide three uniform definitions for bisimulation, trace, and testing equivalences over the ULTraS model. In Sects. 4 to 10, we show that suitable specializations of these three equivalences to the seven specializations of the ULTraS model coincide with behavioral equivalences defined in the literature for the traditional models[1] except when nondeterminism and probability/stochasticity coexist; then new equivalences pop up. Finally, Sect. 11, which contains a table summarizing the considered models and behavioral equivalences, draws some conclusions and discusses future work. For the sake of readability, all the proofs are confined to an appendix.

[1]For each equivalence over traditional models, we will recall only its definition and refer the interested reader to the papers cited at the beginning of the corresponding section for information about congruence properties, equational and logical characterizations, alternative characterizations, and decision algorithms.

2. ULTraS: A Unifying View of LTS-Based Models

The behavior of sequential, concurrent, and distributed processes can be described by means of the so-called labeled transition system (LTS) model [81]. It consists of a set of states, a set of transition labels, and a transition relation. States correspond to the operational modes that processes can pass through. Labels describe the activities that processes can perform independently or in collaboration with each other or with their external environment. The transition relation defines process evolution as determined by the execution of specific activities and is formalized as a *state-to-state* reachability relation.

In this section, we first introduce a generalization of the LTS model that aims at providing a uniform framework that can be employed for defining the behavior of different types of process. In the new model, named ULTraS from Uniform Labeled TRAnsition System, the transition relation associates with any pair composed of a source state and a transition label, a function mapping each possible target state to an element of a preordered set equipped with a minimum. In other words, the state-to-state reachability relation typical of the LTS model is replaced by a *state-to-state-distribution* reachability relation. This is a consequence of the fact that the concept of next state is generalized via a function that *distributes* the one-step reachability among all states by assigning to each possible next state the *degree of reachability* from the current state.

The definition of ULTraS is provided in Sect. 2.1, then we prove that the ULTraS model offers a unifying view of LTS-based models that have been widely used to describe specific classes of processes. More precisely, we consider fully nondeterministic processes in Sect. 2.2, fully probabilistic processes in Sect. 2.3, mixed probabilistic-nondeterministic processes in Sect. 2.4, fully stochastic processes in Sect. 2.5, and mixed stochastic-nondeterministic processes in Sect. 2.6.

2.1. Uniform Labeled Transition Systems

The definition of our uniform model is parameterized with respect to a set D and a preorder relation $\sqsubseteq_D$ on it that is equipped with minimum $\bot_D$. The elements of D are used to express the degree of *one-step* reachability of states, with $\bot_D$ representing unreachability. In the following, we denote by $[S \rightarrow D]$ the set of functions from a set S to D, which is ranged over by $\mathcal{D}$. Whenever S is a set of states, every element $\mathcal{D}$ of $[S \rightarrow D]$ will be interpreted as a *next-state distribution function*.

Definition 2.1. Let $(D, \sqsubseteq_D, \bot_D)$ be a preordered set equipped with a minimum. A uniform labeled transition system on $(D, \sqsubseteq_D, \bot_D)$, or D-ULTraS for short, is a triple $\mathcal{U} = (S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times [S \rightarrow D]$ is a transition relation.

We say that the D-ULTraS $\mathcal{U}$ is functional iff $\longrightarrow$ is a function from $S \times A$ to $[S \rightarrow D]$. ■

Every transition $(s, a, \mathcal{D})$ is written $s \xrightarrow{a} \mathcal{D}$ where $\mathcal{D}(s')$ is a D-value quantifying the degree of reachability of s' from s via that specific a-transition leading to $\mathcal{D}$, with $\mathcal{D}(s') = \bot_D$ meaning that s' is not reachable with that transition. When the considered D-ULTraS is functional, we will write $\mathcal{D}_{s,a}(s')$ to denote the same D-value. We now define the notion of computation for the ULTraS model as a sequence of state-to-state steps, each denoted by $s \stackrel{a}{\longmapsto} s'$ and derived from a state-to-state-distribution transition of the model itself.

Definition 2.2. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS, $n \in \mathbb{N}$, $s_i \in S$ for all $i = 0, \ldots, n$, and $a_i \in A$ for all $i = 1, \ldots, n$. We say that $s_0 \stackrel{a_1}{\longmapsto} s_1 \stackrel{a_2}{\longmapsto} s_2 \ldots s_{n-1} \stackrel{a_n}{\longmapsto} s_n$ is a computation of $\mathcal{U}$ of length n going from s_0 to s_n iff for all $i = 1, \ldots, n$ there exists a transition $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ such that $\mathcal{D}_i(s_i) \neq \bot_D$. ■

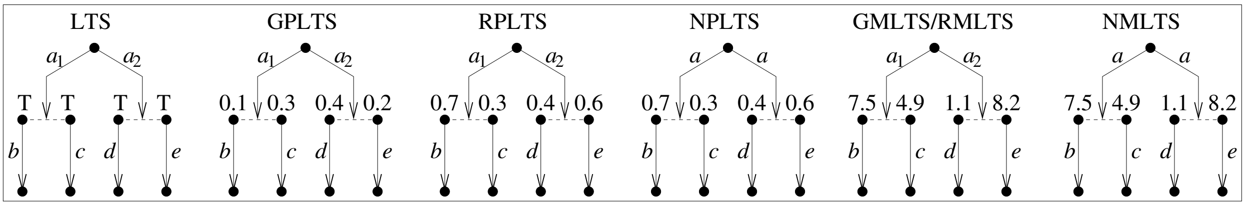

Figure 1: Traditional models represented as ULTRAS models (note that $a_1 \neq a_2$ for functional models)

A D-ULTRAS can be depicted as a directed graph-like structure in which vertices represent states and action-labeled edges represent action-labeled transitions. Given a transition $s \xrightarrow{a} \mathcal{D}$, the corresponding a-labeled edge goes from the vertex representing s to a set of vertices linked by a dashed line, each of which represents a state s' such that $\mathcal{D}(s') \neq \bot_D$ and is labeled with $\mathcal{D}(s')$. Should $\mathcal{D}(s') = \bot_D$ for all states s' – which may happen when the considered D-ULTRAS is functional – the transition would not be depicted at all. Six ULTRAS models are shown in Fig. 1, where the fourth one and the last one are not functional.

2.2. *A Fully Nondeterministic Specialization: The LTS Model*

Fully nondeterministic processes are traditionally represented through state-transition graphs in which every transition is labeled with the action determining the related state change [81]. In these graphs, which correspond to classical automata without final states, there is no information about how to choose among the various transitions departing from a state.

Definition 2.3. A labeled transition system, LTS for short, is a triple $(S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times S$ is a transition relation. ■

Every transition (s, a, s') is written $s \xrightarrow{a} s'$ and means that s can reach s' by executing a.

It is immediate to see that an LTS can be encoded as a functional $\mathbb{B}$-ULTRAS $\mathcal{U}$, where $\mathbb{B} = \{\bot, \top\}$ is the support set of the Boolean algebra with $\bot$ representing false, $\top$ representing true, and $\bot \sqsubseteq_{\mathbb{B}} \top$. Given a transition $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_{s,a}$, $\mathcal{D}_{s,a}(s') = \bot$ means that it is not possible to reach s' from s by executing a, whereas $\mathcal{D}_{s,a}(s') = \top$ means that it is possible.

Definition 2.4. Let $(S, A, \longrightarrow)$ be an LTS. Its corresponding functional $\mathbb{B}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ is defined by letting:

- $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_{s,a}$ for all $s \in S$ and $a \in A$.
- $\mathcal{D}_{s,a}(s') = \begin{cases} \top & \text{if } s \xrightarrow{a} s' \\ \bot & \text{if } (s, a, s') \notin \longrightarrow \end{cases}$ for all $s' \in S$. ■

2.3. *A Fully Probabilistic Specialization: The GPLTS Model*

Fully probabilistic processes, also called generative probabilistic processes according to the terminology of [125], can be represented through state-transition graphs in which every transition is labeled with both the action and the probability of the related state change. In other words, each such process corresponds to an action-labeled discrete-time Markov chain (ADTMC), i.e., a discrete-time Markov chain [118] whose transitions are additionally labeled with actions.[2]

In the following, we use $\{\!|$ and $|\!\}$ to delimit multisets. We also assume that the summation over the empty multiset of numbers is zero.

Definition 2.5. A generative probabilistic labeled transition system, GPLTS for short, is a triple $(S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times \mathbb{R}_{(0,1]} \times S$ is a transition relation.
- For all $s, s' \in S$ and $a \in A$, whenever $(s, a, p_1, s'), (s, a, p_2, s') \in \longrightarrow$, then $p_1 = p_2$.
- For all $s \in S$, it holds that $\sum \{\!| p \in \mathbb{R}_{(0,1]} \mid \exists a \in A. \exists s' \in S. (s, a, p, s') \in \longrightarrow |\!\} \in \{0, 1\}$. ■

Every transition (s, a, p, s') is written $s \xrightarrow{a,p} s'$, with p being the probability with which s' is reached from s by executing a.

It is easy to see that a GPLTS can be encoded as a functional $\mathbb{R}_{[0,1]}$-ULTRAS $\mathcal{U}$ in which $\sum_{a \in A} \sum_{s' \in S} \mathcal{D}_{s,a}(s') \in \{0, 1\}$ for all $s \in S$, where $\sqsubseteq_{\mathbb{R}_{[0,1]}}$ is the usual ordering of reals whose minimum is 0. Given a transition $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_{s,a}$, $\mathcal{D}_{s,a}(s') = 0$ means that it is not possible to reach s' from s by executing a, whereas $\mathcal{D}_{s,a}(s') \in \mathbb{R}_{(0,1]}$ means that it is possible with probability $p = \mathcal{D}_{s,a}(s')$.

Definition 2.6. Let $(S, A, \longrightarrow)$ be a GPLTS. Its corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ is defined by letting:

- $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_{s,a}$ for all $s \in S$ and $a \in A$.
- $\mathcal{D}_{s,a}(s') = \begin{cases} p & \text{if } s \xrightarrow{a,p} s' \\ 0 & \text{if } \nexists p \in \mathbb{R}_{(0,1]}.\, s \xrightarrow{a,p} s' \end{cases}$ for all $s' \in S$. ■

2.4. Two Mixed Probabilistic-Nondeterministic Specializations: The RPLTS and NPLTS Models

In an LTS-based model, probability and nondeterminism can be combined in at least two different ways, which give rise to two variants of Markov decision processes (MDP) [56].

In the first case, assuming that transitions are labeled with both actions and probabilities, unlike fully probabilistic processes the probabilities can be enforced only among transitions departing from the same state that are labeled with the same action. In other words, the choice among transitions labeled with the same action is probabilistic, whereas the choice among transitions labeled with different actions is nondeterministic. According to the terminology of [125], the resulting processes are reactive probabilistic processes and correspond to probabilistic automata in the sense of [107].

Definition 2.7. A reactive probabilistic labeled transition system, RPLTS for short, is a triple $(S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.

[2]The name discrete-time Markov chain is used here for historical reasons. Since time does not really come into play, a name like time-abstract Markov chain would be better. A discrete-time interpretation is appropriate only when all state changes occur at equidistant time points.

- $\longrightarrow \subseteq S \times A \times \mathbb{R}_{(0,1]} \times S$ is a transition relation.
- For all $s, s' \in S$ and $a \in A$, whenever $(s, a, p_1, s'), (s, a, p_2, s') \in \longrightarrow$, then $p_1 = p_2$.
- For all $s \in S$ and $a \in A$, it holds that $\sum \{\!| p \in \mathbb{R}_{(0,1]} \mid \exists s' \in S. (s, a, p, s') \in \longrightarrow |\!\} \in \{0, 1\}$. ■

The difference between a GPLTS and an RPLTS has to do with the last constraint imposed on the transition relation. Given a state with outgoing transitions, in a GPLTS the probabilities of all those transitions must sum up to one, while in an RPLTS this must be the case for each maximal subset of transitions among the considered ones that are labeled with the same action. As a consequence, given a transition $s \xrightarrow{a,p} s'$ of an RPLTS, the value p is the probability with which s' is reached from s by executing a *conditioned* on the fact that a has been chosen among all the possible actions. Therefore, an RPLTS can be encoded as a functional $\mathbb{R}_{[0,1]}$-ULTRAS in which $\sum_{s' \in S} \mathcal{D}_{s,a}(s') \in \{0, 1\}$ for all $s \in S$ and $a \in A$, where again $\sqsubseteq_{\mathbb{R}_{[0,1]}}$ is the usual ordering of reals whose minimum is 0.

Definition 2.8. Let $(S, A, \longrightarrow)$ be an RPLTS. Its corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ is defined by letting:

- $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_{s,a}$ for all $s \in S$ and $a \in A$.
- $\mathcal{D}_{s,a}(s') = \begin{cases} p & \text{if } s \xrightarrow{a,p} s' \\ 0 & \text{if } \nexists p \in \mathbb{R}_{(0,1]}.\, s \xrightarrow{a,p} s' \end{cases}$ for all $s' \in S$. ■

In the second case, also the choice among transitions labeled with the same action is nondeterministic. In this setting, transitions are labeled only with actions while probabilities are embedded in a next-state distribution function, so that the transition relation becomes a state-to-state-distribution reachability relation. The resulting processes are nondeterministic and probabilistic processes that are representative of a number of slightly different probabilistic computational models including internal nondeterminism among which we mention concurrent (discrete-time) Markov chains [127], alternating probabilistic models [65, 129, 103], probabilistic automata in the sense of [109], and the denotational probabilistic models in [75] (see [115] for an overview).

Definition 2.9. A nondeterministic and probabilistic labeled transition system, NPLTS for short, is a triple $(S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times [S \to \mathbb{R}_{[0,1]}]$ is a transition relation.
- For all $(s, a, \mathcal{D}) \in \longrightarrow$, it holds that $\sum_{s' \in S} \mathcal{D}(s') = 1$. ■

Obviously, an NPLTS is an $\mathbb{R}_{[0,1]}$-ULTRAS in which $\sum_{s' \in S} \mathcal{D}(s') = 1$ for all $(s, a, \mathcal{D}) \in \longrightarrow$, where once more $\sqsubseteq_{\mathbb{R}_{[0,1]}}$ is the usual ordering of reals whose minimum is 0. Notice that, unlike all the previous specializations, such an ULTRAS is not necessarily functional due to the coexistence of internal nondeterminism and probabilistic choices.

2.5. A Fully Stochastic Specialization: The GMLTS Model

Fully stochastic processes in which the notion of time is formalized by means of exponentially distributed durations, also called (generative) Markovian processes, can be represented through state-transition graphs in which every transition is labeled with both the action and the rate of the related state change. In other words, each such process corresponds to an action-labeled continuous-time Markov chain (ACTMC), i.e., a continuous-time Markov chain [118] whose transitions are additionally labeled with actions.

Any ACTMC can be viewed as being obtained from an ADTMC in which every state s is labeled with an exponentially distributed sojourn time, which is uniquely identified by a positive real number $\mathrm{E}(s)$ called state exit rate, whose reciprocal coincides with the average sojourn time in s. In fact, if we assume that transition firing is governed by a race policy – i.e., the transition that is chosen in every state is the one sampling the least duration – then in this ADTMC augmented with state exit rates we can merge probabilistic and time information by eliminating all state labels and replacing the probability labeling each transition departing from any state s with a rate given by $\mathrm{E}(s)$ multiplied by the transition probability. The sum of the resulting transition rates is equal to $\mathrm{E}(s)$, which is consistent with the adoption of the race policy and the fact that the minimum of a set of exponentially distributed random variables is exponentially distributed with rate equal to the sum of the original rates.

Definition 2.10. A generative Markovian labeled transition system, GMLTS for short, is a triple $(S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times \mathbb{R}_{>0} \times S$ is a transition relation.
- For all $s, s' \in S$ and $a \in A$, whenever $(s, a, \lambda_1, s'), (s, a, \lambda_2, s') \in \longrightarrow$, then $\lambda_1 = \lambda_2$. ■

Every transition (s, a, λ, s') is written $s \xrightarrow{a,\lambda} s'$, with λ being the rate at which s' is reached from s by executing a and hence $1/\lambda$ being the average duration of the transition.

It is straightforward to see that a GMLTS can be encoded as a functional $\mathbb{R}_{\geq 0}$-ULTraS $\mathcal{U}$, where $\sqsubseteq_{\mathbb{R}_{\geq 0}}$ is the usual ordering of reals whose minimum is 0. Given a transition $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_{s,a}$, $\mathcal{D}_{s,a}(s') = 0$ means that it is not possible to reach s' from s by executing a, whereas $\mathcal{D}_{s,a}(s') \in \mathbb{R}_{>0}$ means that it is possible at rate $\lambda = \mathcal{D}_{s,a}(s')$. Note that $\mathrm{E}(s) = \sum_{a \in A} \sum_{s' \in S} \mathcal{D}_{s,a}(s')$ for all $s \in S$.

Definition 2.11. Let $(S, A, \longrightarrow)$ be a GMLTS. Its corresponding functional $\mathbb{R}_{\geq 0}$-ULTraS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ is defined by letting:

- $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_{s,a}$ for all $s \in S$ and $a \in A$.
- $\mathcal{D}_{s,a}(s') = \begin{cases} \lambda & \text{if } s \xrightarrow{a,\lambda} s' \\ 0 & \text{if } \nexists \lambda \in \mathbb{R}_{>0}.\, s \xrightarrow{a,\lambda} s' \end{cases}$ for all $s' \in S$. ■

2.6. Two Mixed Stochastic-Nondeterministic Specializations: The RMLTS and NMLTS Models

Similar to the probabilistic case, in an LTS-based model stochasticity and nondeterminism can be combined in at least two different ways, which give rise to two variants of continuous-time Markov decision processes (CTMDP) [106].

In the first case, assuming that transitions are labeled with both actions and rates, unlike fully stochastic processes the race policy can be enforced only among transitions departing from the same state that are labeled with the same action. In other words, the choice among transitions labeled with the same action is governed by the race policy, whereas the choice among transitions labeled with different actions is nondeterministic. As a consequence, only the conditional exit rate $\mathrm{E}_a(s)$ can be defined, which is the sum of the rates of the transitions departing from state s that are labeled with action a. The resulting processes are reactive Markovian processes and correspond to continuous-time probabilistic automata [82].

Definition 2.12. A reactive Markovian labeled transition system, RMLTS for short, is a triple $(S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times \mathbb{R}_{>0} \times S$ is a transition relation.
- For all $s, s' \in S$ and $a \in A$, whenever $(s, a, \lambda_1, s'), (s, a, \lambda_2, s') \in \longrightarrow$, then $\lambda_1 = \lambda_2$. ■

The difference between a GMLTS and an RMLTS is simply related to the scope of the race policy: it is generative in the former model, whereas it is reactive in the latter model. As a consequence, given a transition $s \xrightarrow{a,\lambda} s'$ of an RMLTS, the value λ is the rate at which s' is reached from s by executing a *conditioned* on the fact that a has been chosen among all the possible actions. Therefore, an RMLTS can be encoded as a functional $\mathbb{R}_{\geq 0}$-ULTraS, where again $\sqsubseteq_{\mathbb{R}_{\geq 0}}$ is the usual ordering of reals whose minimum is 0. Note that $\mathrm{E}_a(s) = \sum_{s' \in S} \mathcal{D}_{s,a}(s')$ for all $s \in S$ and $a \in A$.

Definition 2.13. Let $(S, A, \longrightarrow)$ be an RMLTS. Its corresponding functional $\mathbb{R}_{\geq 0}$-ULTraS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ is defined by letting:

- $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_{s,a}$ for all $s \in S$ and $a \in A$.
- $\mathcal{D}_{s,a}(s') = \begin{cases} \lambda & \text{if } s \xrightarrow{a,\lambda} s' \\ 0 & \text{if } \nexists \lambda \in \mathbb{R}_{>0}.\, s \xrightarrow{a,\lambda} s' \end{cases}$ for all $s' \in S$. ■

In the second case, also the choice among transitions labeled with the same action is nondeterministic. In this setting, transitions are labeled only with actions while rates are embedded in a next-state distribution function, so that the transition relation becomes a state-to-state-distribution reachability relation. The resulting processes are nondeterministic and Markovian processes.

Definition 2.14. A nondeterministic and Markovian labeled transition system, NMLTS for short, is a triple $(S, A, \longrightarrow)$ where:

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times [S \to \mathbb{R}_{\geq 0}]$ is a transition relation.
- For all $(s, a, \mathcal{D}) \in \longrightarrow$, it holds that $\sum_{s' \in S} \mathcal{D}(s') > 0$. ■

Obviously, an NMLTS is an $\mathbb{R}_{\geq 0}$-ULTraS in which $\sum_{s' \in S} \mathcal{D}(s') > 0$ for all $(s, a, \mathcal{D}) \in \longrightarrow$, where once more $\sqsubseteq_{\mathbb{R}_{\geq 0}}$ is the usual ordering of reals whose minimum is 0. However, it is not necessarily functional due to the coexistence of internal nondeterminism and rate-based probabilistic choices.

3. Bisimulation, Trace, and Testing Equivalences for the ULTraS Model

LTS-based models come equipped with equivalences through which it is possible to compare processes on the basis of their behavior and reduce their state spaces before analyzing their properties. These behavioral equivalences result in a linear-time/branching-time spectrum [46, 124, 79, 74, 11, 3] including several variants of three major approaches: bisimulation [68], trace [31], and testing [47].

In this section, we show that bisimulation, trace, and testing equivalences can be defined in a uniform manner over the ULTraS model, thus emphasizing the adequacy of this model as a unifying semantic framework. In the subsequent sections, we will see that suitable instances of the uniform definition of the three equivalences coincide with the corresponding behavioral equivalences defined in the literature in the case of the seven classes of processes considered in Sect. 2, except when internal nondeterminism and probability/stochasticity coexist.

3.1. Multi-Step Reachability Measure Functions

The definition of bisimulation, trace, and testing equivalences over the ULTRAS model is parameterized with respect to a measure function that expresses the degree of *multi-step* reachability of a set of states. Similar to the one-step reachability encoded within an ULTRAS, in which we consider individual actions, multi-step reachability has to do with sequences of actions commonly called traces, which are the observable effects of the computations performed by an ULTRAS.

Definition 3.1. Let A be a countable set of transition-labeling actions. A trace α is an element of A^*, where $\alpha = \varepsilon$ denotes the empty trace, operation "$|\_|$" computes the length of a trace, and operation "$\_ \circ \_$" computes the concatenation of two traces. ■

Definition 3.2. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS and $(M, \sqsubseteq_M, \bot_M)$ be a preordered set equipped with a minimum. A measure function on $(M, \sqsubseteq_M, \bot_M)$ for $\mathcal{U}$, or M-measure function for $\mathcal{U}$ for short, is a function $\mathcal{M}_M : S \times A^* \times 2^S \to M$ such that the value of $\mathcal{M}_M(s, \alpha, S')$ is defined by induction on $|\alpha|$ and depends only on the reachability of a state in S' from state s through computations labeled with trace α. ■

From a conceptual viewpoint, a measure function subsumes the existence of two operators. The first operator, which we may think of as an additive operator $\oplus$, determines the M-value representing the reachability of a state in S' from state s via trace α by combining the M-values representing the reachability of a state in S' from state s along each single computation labeled with trace α that goes from s to S'. The second operator, which we may think of as a multiplicative operator $\otimes$, determines the M-value representing the reachability of a state in S' from state s along a single computation labeled with trace α that goes from s to S' by combining the D-values representing reachability at each individual step of the computation.

Note that different measure functions can induce different variants of a behavioral equivalence on the same D-ULTRAS depending on the choice of $(M, \sqsubseteq_M, \bot_M)$. Although D and M may be the same support set, this is not necessarily the case as we will see later on. In fact, while a D-value is related to one-step reachability, an M-value – of the form $\mathcal{M}_M(s, \alpha, S')$ – is computed on the basis of D-values to quantify multi-step reachability, with $\bot_M$ representing multi-step unreachability. For instance, in the testing equivalence for fully nondeterministic processes, the M-value will be a *pair* of $\mathbb{B}$-values – rather than a single $\mathbb{B}$-value – expressing the possibility and the necessity of reaching S' from s after α. For probabilistic processes including internal nondeterminism, it will be a nonemtpy *set* of $\mathbb{R}_{[0,1]}$-values – rather than a single $\mathbb{R}_{[0,1]}$-value – containing the probability of reaching S' from s after α for each possible way of resolving nondeterminism. For stochastic processes, it will be an $\mathbb{R}_{[0,1]}$-valued (or in case of internal nondeterminism a $2^{\mathbb{R}_{[0,1]}}$-valued) *function* – rather than a single $\mathbb{R}_{\geq 0}$-value – representing for each possible deadline the probability of reaching S' from s after α within the considered deadline (for each possible way of resolving nondeterminism).

3.2. A Uniform Definition of Bisimulation Equivalence

The basic idea behind bisimulation equivalence is that of capturing the ability of two models of mimicking each other's behavior at every step. In the literature, two different ways of defining a bisimulation relation have been given. In the case of fully nondeterministic models, for any pair of related states it is sufficient to compare the individual states reached by performing equally labeled transitions. In the case of quantitative models, equivalence classes of reachable states are considered, because it is necessary to combine probabilities or rates across equally labeled transitions departing from the same state that reach equivalent states.

For ULTRAS models, we provide an even more general definition that examines groups of equivalence classes. The objective is to adequately support models in which nondeterminism and quantitative aspects coexist by opening the possibility that, during the bisimulation game, a single transition on one side is matched by several equally labeled transitions on the other side with respect to different sets of states. We note that the more traditional definition based on equivalence classes is easily reobtained by restricting attention to singleton groups. Given a D-ULTRAS $\mathcal{U} = (S, A, \longrightarrow)$, a state $s \in S$, an action $a \in A$, a group $\mathcal{G} \in 2^{S/\mathcal{B}}$ of equivalence classes with respect to an equivalence relation $\mathcal{B}$ over S, and an M-measure function $\mathcal{M}_M$ for $\mathcal{U}$, the M-values that we need to compare in the case of bisimulation equivalence are thus of the form $\mathcal{M}_M(s, a, \bigcup \mathcal{G})$, where $\bigcup \mathcal{G}$ is the union of all the equivalence classes in $\mathcal{G}$.

Definition 3.3. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS and $\mathcal{M}_M$ be an M-measure function for $\mathcal{U}$. An equivalence relation $\mathcal{B}$ over S is an $\mathcal{M}_M$-bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_M(s_1, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_M(s_2, a, \bigcup \mathcal{G})$$

We say that $s_1, s_2 \in S$ are $\mathcal{M}_M$-bisimilar, written $s_1 \sim_{\mathrm{B},\mathcal{M}_M} s_2$, iff there exists an $\mathcal{M}_M$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

3.3. A Uniform Definition of Trace Equivalence

Trace equivalence compares the ability of two models of performing equally labeled computations thereby abstracting from branching points in their behavior. With respect to bisimulation equivalence, entire traces have to be considered instead of individual actions, whilst the set of destination states is not important. Given a D-ULTRAS $\mathcal{U} = (S, A, \longrightarrow)$, a state $s \in S$, a trace $\alpha \in A^*$, and an M-measure function $\mathcal{M}_M$ for $\mathcal{U}$, the M-values that we need to compare in the case of trace equivalence are thus of the form $\mathcal{M}_M(s, \alpha, S)$.

Definition 3.4. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS and $\mathcal{M}_M$ be an M-measure function for $\mathcal{U}$. We say that $s_1, s_2 \in S$ are $\mathcal{M}_M$-trace equivalent, written $s_1 \sim_{\mathrm{Tr},\mathcal{M}_M} s_2$, iff for all traces $\alpha \in A^*$:

$$\mathcal{M}_M(s_1, \alpha, S) \;=\; \mathcal{M}_M(s_2, \alpha, S)$$

■

3.4. A Uniform Definition of Testing Equivalence

The definition of testing equivalence requires the formalization of the notion of test and the consideration of configurations rather than ordinary states. A test specifies which actions of a process are permitted at each step and can be expressed as a suitable D-ULTRAS that includes a success state, which determines the computations that are successful.[3] If this D-ULTRAS is not functional, then the success state cannot have outgoing transitions, otherwise the next-state distribution function of each transition departing from the success state must be identically equal to $\bot_D$. For the sake of simplicity, we restrict ourselves to finite tests, i.e., tests whose underlying graph structure is finite state and acyclic – so that only finite-length computations are considered – and finitely branching – so that only a choice among finitely many alternative actions is made available at each step.

Definition 3.5. Let $(D, \sqsubseteq_D, \bot_D)$ be a preordered set equipped with a minimum. An observation system on $(D, \sqsubseteq_D, \bot_D)$, or D-observation system for short, is a finite-state, acyclic, and finitely-branching D-ULTRAS $\mathcal{O} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω such that, whenever $\omega \xrightarrow{a} \mathcal{D}$, then $\mathcal{D}(o) = \bot_D$ for all $o \in O$. We say that a computation of $\mathcal{O}$ is successful iff its last state is ω. ■

A D-ULTRAS can be tested only through a D-observation system by running them in parallel and enforcing synchronization on any action. The states of the resulting D-ULTRAS are called configurations and are pairs each formed by a state of the D-ULTRAS under test and a state of the D-observation system. A configuration can evolve to a new configuration only through the synchronization of two transitions – departing from the two states constituting the configuration – that are labeled with the same action.

For each such pair of synchronizing transitions, the next-state distribution function of the resulting transition is obtained from the two original next-state distribution functions by means of a suitable D-valued function δ, which computes the degree of one-step reachability of every possible target configuration. Since $\bot_D$ represents unreachability, function δ must be $\bot_D$-preserving, i.e., it must yield $\bot_D$ iff at least one of its arguments is $\bot_D$. As a consequence of this first constraint, in the case of nondeterministic processes δ boils down to logical conjunction, whereas several alternative options are available in the case of probabilistic and stochastic processes. In order for tests not to blur distinctions among processes, function δ must also be injective over tuples of D-values not including $\bot_D$ when considering one

[3]Unlike the original testing theory of [47] in which a success action is used, here we employ a success state as it is more convenient from a technical viewpoint when dealing with quantitative domains.

argument at a time. This second constraint can be formalized as follows: given $n \in \mathbb{N}_{\geq 2}$ and two tuples $(d_1, \ldots, d_{i-1}, d', d_{i+1}, \ldots, d_n), (d_1, \ldots, d_{i-1}, d'', d_{i+1}, \ldots, d_n) \in (D \setminus \{\bot_D\})^n$, if $\delta : D^n \to D$ and $d' \neq d''$ then $\delta(d_1, \ldots, d_{i-1}, d', d_{i+1}, \ldots, d_n) \neq \delta(d_1, \ldots, d_{i-1}, d'', d_{i+1}, \ldots, d_n)$. For the sake of conciseness, we say that δ must be $\bot_D$-preserving and argument-injective. Note that n can be greater than 2 in order to take into account further information such as normalizing factors commonly used for probabilistic processes.

Definition 3.6. Let $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be a D-ULTRAS, $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ be a D-observation system, and δ be a $\bot_D$-preserving and argument-injective D-valued function. The interaction system of $\mathcal{U}$ and $\mathcal{O}$ with respect to δ is the D-ULTRAS $\mathcal{I}^\delta(\mathcal{U}, \mathcal{O}) = (S \times O, A, \longrightarrow)$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$. We denote by $\mathcal{S}^\delta(\mathcal{U}, \mathcal{O})$ the set of successful configurations of $\mathcal{I}^\delta(\mathcal{U}, \mathcal{O})$.

- The transition relation $\longrightarrow \subseteq (S \times O) \times A \times [(S \times O) \to D]$ is such that $(s, o) \xrightarrow{a} \mathcal{D}$ iff $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_1$ and $o \xrightarrow{a}_{\mathcal{O}} \mathcal{D}_2$ with $\mathcal{D}(s', o')$ being obtained from $\mathcal{D}_1(s')$ and $\mathcal{D}_2(o')$ by applying δ. We say that a computation of $\mathcal{I}^\delta(\mathcal{U}, \mathcal{O})$ is successful iff its last configuration is successful.

The extension to $\mathcal{I}^\delta(\mathcal{U}, \mathcal{O})$ of an M-measure function $\mathcal{M}_M$ for $\mathcal{U}$ is defined as the M-measure function $\mathcal{M}_M^{\delta,\mathcal{O}} : (S \times O) \times A^* \times 2^{S \times O} \to M$ obtained from $\mathcal{M}_M$ by replacing states and transitions of $\mathcal{U}$ with configurations and transitions of $\mathcal{I}^\delta(\mathcal{U}, \mathcal{O})$. ■

Two states of $\mathcal{U}$ are $\mathcal{M}_M^\delta$-testing equivalent with respect to $\mathcal{O}$ iff, for each trace α, according to $\mathcal{M}_M$ the two configurations respectively including the two states and the initial state of $\mathcal{O}$ result in the same δ-based degree of multi-step reachability via α towards the set of successful configurations. Therefore, given a state s and indicating with o the initial state of $\mathcal{O}$, the M-values that we need to compare in the case of testing equivalence are of the form $\mathcal{M}_M^{\delta,\mathcal{O}}((s, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$.

Definition 3.7. Let $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be a D-ULTRAS, $\mathcal{M}_M$ be an M-measure function for $\mathcal{U}$, and δ be a $\bot_D$-preserving and argument-injective D-valued function. We say that $s_1, s_2 \in S$ are $\mathcal{M}_M^\delta$-testing equivalent, written $s_1 \sim_{\mathrm{Te}, \mathcal{M}_M^\delta} s_2$, iff for every D-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all traces $\alpha \in A^*$:

$$\mathcal{M}_M^{\delta,\mathcal{O}}((s_1, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O})) \;=\; \mathcal{M}_M^{\delta,\mathcal{O}}((s_2, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$$ ■

3.5. Inclusion Relations Among the Three Uniform Equivalences

We conclude by showing that, as expected, bisimulation equivalence for ULTRAS models is finer than testing equivalence for ULTRAS models, which in turn is finer than trace equivalence for ULTRAS models. Of course, this holds true when fixing the two preordered sets $(D, \sqsubseteq_D, \bot_D)$ for one-step reachability in the models and $(M, \sqsubseteq_M, \bot_M)$ for multi-step reachability in the equivalences.

Theorem 3.8. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $\mathcal{M}_M$ be an M-measure function for $\mathcal{U}$, and δ be a $\bot_D$-preserving and argument-injective D-valued function. Then for all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{B}, \mathcal{M}_M} s_2 \implies s_1 \sim_{\mathrm{Te}, \mathcal{M}_M^\delta} s_2 \implies s_1 \sim_{\mathrm{Tr}, \mathcal{M}_M} s_2$$ ■

4. Equivalences for Fully Nondeterministic Processes

In this section, we instantiate the three behavioral equivalences of Sect. 3 – i.e., bisimulation, trace, and testing equivalences – for fully nondeterministic processes represented as functional $\mathbb{B}$-ULTRAS models (see Sect. 2.2). This is accomplished by introducing two measure functions based on logical disjunction and logical conjunction, respectively, each associating a suitable $\mathbb{B}$-value with every triple composed of a source state s, a trace α, and a set of destination states S'. More precisely, the first function computes a value $\mathcal{M}_{\mathbb{B},\vee}(s, \alpha, S')$ that establishes whether there exists a computation that is labeled with trace α and leads to a state in S' from state s. By contrast, the second function computes a value $\mathcal{M}_{\mathbb{B},\wedge}(s, \alpha, S')$ that

$\mathcal{M}_{\mathbb{B},\vee}(s,\alpha,S') \;=\; \begin{cases} \bigvee\limits_{s' \in S \text{ s.t. } \mathcal{D}_{s,a}(s') \neq \bot} \mathcal{M}_{\mathbb{B},\vee}(s',\alpha',S') & \text{if } \alpha = a \circ \alpha' \text{ and } \exists s' \in S.\, \mathcal{D}_{s,a}(s') \neq \bot \\ \top & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ \bot & \text{if } \alpha = a \circ \alpha' \text{ and } \nexists s' \in S.\, \mathcal{D}_{s,a}(s') \neq \bot \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$
$\mathcal{M}_{\mathbb{B},\wedge}(s,\alpha,S') \;=\; \begin{cases} \bigwedge\limits_{s' \in S \text{ s.t. } \mathcal{D}_{s,a}(s') \neq \bot} \mathcal{M}_{\mathbb{B},\wedge}(s',\alpha',S') & \text{if } \alpha = a \circ \alpha' \text{ and } \exists s' \in S.\, \mathcal{D}_{s,a}(s') \neq \bot \\ \top & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ \bot & \text{if } \alpha = a \circ \alpha' \text{ and } \nexists s' \in S.\, \mathcal{D}_{s,a}(s') \neq \bot \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$
$\mathcal{M}_{\mathbb{B}\times\mathbb{B}}(s,\alpha,S') \;=\; (\mathcal{M}_{\mathbb{B},\vee}(s,\alpha,S'), \mathcal{M}_{\mathbb{B},\wedge}(s,\alpha,S'))$

Table 1: Measure functions for functional $\mathbb{B}$-ULTRAS models representing fully nondeterministic processes

establishes whether there exists a computation that is labeled with trace α and leads to a state in S' from state s and, in such a case, whether any computation from state s labeled with a prefix of trace α is part of a computation from state s that is labeled with the entire trace α and leads to a state in S'. In other words, the two measure functions respectively express the possibility and the necessity of reaching S' from s through α. We then introduce a third measure function that combines the first two functions.

Definition 4.1. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{B}$-ULTRAS:

- The measure functions $\mathcal{M}_{\mathbb{B},\vee} : S \times A^* \times 2^S \to \mathbb{B}$ and $\mathcal{M}_{\mathbb{B},\wedge} : S \times A^* \times 2^S \to \mathbb{B}$ for $\mathcal{U}$ are inductively defined in the first two parts of Table 1, where the associated preorder relation is the ordering $\sqsubseteq_{\mathbb{B}}$ of Booleans whose minimum is $\bot$.
- The measure function $\mathcal{M}_{\mathbb{B}\times\mathbb{B}} : S \times A^* \times 2^S \to \mathbb{B}\times\mathbb{B}$ for $\mathcal{U}$ is defined in the third part of Table 1, where the associated preorder relation, whose minimum is $(\bot,\bot)$, is defined by letting $(v_{1,1}, v_{1,2}) \sqsubseteq_{\mathbb{B}\times\mathbb{B}} (v_{2,1}, v_{2,2})$ iff $v_{1,1} \sqsubseteq_{\mathbb{B}} v_{2,1}$ and $v_{1,2} \sqsubseteq_{\mathbb{B}} v_{2,2}$. ■

Note that the first clause of $\mathcal{M}_{\mathbb{B},\vee}(s,\alpha,S')$ could have been defined as $\bigvee_{s' \in S} \mathcal{D}_{s,a}(s') \wedge \mathcal{M}_{\mathbb{B},\vee}(s',\alpha',S')$ while the first clause of $\mathcal{M}_{\mathbb{B},\wedge}(s,\alpha,S')$ could *not* have been defined as $\bigwedge_{s' \in S} \mathcal{D}_{s,a}(s') \wedge \mathcal{M}_{\mathbb{B},\wedge}(s',\alpha',S')$ because, in the presence of at least one state $s' \in S$ such that $\mathcal{D}_{s,a}(s') = \bot$, we would erroneously get $\bot$.

We now show that the three resulting behavioral equivalences $\sim_{\mathrm{B},\mathcal{M}_{\mathbb{B},\vee}}$, $\sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{B},\vee}}$, and $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{LC}}_{\mathbb{B}\times\mathbb{B}}}$ on functional $\mathbb{B}$-ULTRAS models – where LC stands for logical conjunction, the only $\bot$-preserving and argument-injective $\mathbb{B}$-valued function – respectively coincide with the bisimulation, trace, and testing equivalences defined in the literature for LTS models.

4.1. Bisimulation Equivalence

Bisimilarity for LTS models [100, 68, 90, 14, 80, 98] compares the ability of two fully nondeterministic processes of mimicking each other's behavior at every step.

Definition 4.2. Let $(S, A, \longrightarrow)$ be an LTS. A binary relation $\mathcal{B}$ over S is a bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$:

- Whenever $s_1 \xrightarrow{a} s'_1$, then $s_2 \xrightarrow{a} s'_2$ with $(s'_1, s'_2) \in \mathcal{B}$.
- Whenever $s_2 \xrightarrow{a} s'_2$, then $s_1 \xrightarrow{a} s'_1$ with $(s'_1, s'_2) \in \mathcal{B}$.

We say that $s_1, s_2 \in S$ are bisimilar, written $s_1 \sim_{\mathrm{B}} s_2$, iff there exists a bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

Theorem 4.3. Let $(S, A, \longrightarrow)$ be an LTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{B}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{B}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathbb{B},\vee}} s_2 \qquad \blacksquare$$

4.2. Trace Equivalence

Similar to language equivalence for ordinary automata in the case that all the states are accepting, trace equivalence for LTS models [31] compares the ability of two fully nondeterministic processes of performing computations labeled with the same traces. In order to formalize this for an LTS $(S, A, \longrightarrow)$, we need to lift the transition relation from actions to action sequences by letting $s \xRightarrow{a_1 \ldots a_n} s' \equiv s \xrightarrow{a_1} s_1 \ldots s_{n-1} \xrightarrow{a_n} s'$ when $n \in \mathbb{N}_{>0}$ and $s \xRightarrow{\varepsilon} s$ when $n = 0$. Given $s \in S$ and $\alpha \in A^*$, we also write $s \xRightarrow{\alpha}$ to denote the existence of $s' \in S$ such that $s \xRightarrow{\alpha} s'$, i.e., the existence of a computation from s labeled with α.

Definition 4.4. Let $(S, A, \longrightarrow)$ be an LTS. We say that $s_1, s_2 \in S$ are trace equivalent, written $s_1 \sim_{\mathrm{Tr}} s_2$, iff for all traces $\alpha \in A^*$:

- If $s_1 \xRightarrow{\alpha}$, then $s_2 \xRightarrow{\alpha}$.
- If $s_2 \xRightarrow{\alpha}$, then $s_1 \xRightarrow{\alpha}$. ■

Theorem 4.5. Let $(S, A, \longrightarrow)$ be an LTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{B}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{Tr}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{B},\vee}} s_2 \qquad \blacksquare$$

4.3. Testing Equivalence

Testing equivalence for LTS models [47, 66, 46, 39] compares two fully nondeterministic processes on the basis of the fact that they may or must pass the same tests. The idea is that each of the two processes is run in parallel with any test by enforcing synchronization on any action, then the responses to the test provided by the two processes are compared. Tests are formalized as LTS models equipped with a success state.

Definition 4.6. A fully nondeterministic test is a finite-state, acyclic, and finitely-branching LTS $\mathcal{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathcal{T}$ is successful iff its last state is ω. ■

Definition 4.7. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be an LTS and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a fully nondeterministic test. The interaction system of $\mathcal{L}$ and $\mathcal{T}$ is the LTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = (S \times O, A, \longrightarrow)$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.
- The transition relation $\longrightarrow \subseteq (S \times O) \times A \times (S \times O)$ is such that $(s, o) \xrightarrow{a} (s', o')$ iff $s \xrightarrow{a}_{\mathcal{L}} s'$ and $o \xrightarrow{a}_{\mathcal{T}} o'$. We say that a computation of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ is successful iff its last configuration is successful. ■

Definition 4.8. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be an LTS, $s \in S$, and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a fully nondeterministic test with initial state $o \in O$. We say that:

- s may pass $\mathcal{T}$ iff there exists at least one successful computation in $\mathcal{I}(\mathcal{L}, \mathcal{T})$ from (s, o).
- s must pass $\mathcal{T}$ iff all maximal computations in $\mathcal{I}(\mathcal{L}, \mathcal{T})$ from (s, o) are successful. ■

Definition 4.9. Let $(S, A, \longrightarrow)$ be an LTS. We say that $s_1, s_2 \in S$ are testing equivalent, written $s_1 \sim_{\mathrm{Te}} s_2$, iff for every fully nondeterministic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$s_1 \text{ may pass } \mathcal{T} \iff s_2 \text{ may pass } \mathcal{T}$$
$$s_1 \text{ must pass } \mathcal{T} \iff s_2 \text{ must pass } \mathcal{T} \qquad \blacksquare$$

$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S') \;=\; \begin{cases} \sum\limits_{s' \in S} \mathcal{D}_{s,a}(s') \cdot \mathcal{M}_{\mathbb{R}_{[0,1]}}(s', \alpha', S') & \text{if } \alpha = a \circ \alpha' \\ 1 & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ 0 & \text{if } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$

Table 2: Measure function for functional $\mathbb{R}_{[0,1]}$-ULTraS models representing generative or reactive probabilistic processes

Consistent with Def. 4.7, we denote by LC : $\mathbb{B} \times \mathbb{B} \to \mathbb{B}$ the logical conjunction function defined by letting $\text{LC}(v_1, v_2) = v_1 \wedge v_2$, which is the only $\perp$-preserving and argument-injective $\mathbb{B}$-valued function. Given a $\mathbb{B}$-ULTraS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ and a $\mathbb{B}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$, for all configurations (s, o) of their interaction system $\mathcal{I}^{\text{LC}}(\mathcal{U}, \mathcal{O})$ and for all $a \in A$ we let $(s, o) \xrightarrow{a} \mathcal{D}$ iff $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_1$ and $o \xrightarrow{a}_{\mathcal{O}} \mathcal{D}_2$ with $\mathcal{D}(s', o') = \text{LC}(\mathcal{D}_1(s'), \mathcal{D}_2(o'))$ for each $(s', o') \in S \times O$.

Theorem 4.10. Let $(S, A, \longrightarrow)$ be an LTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{B}$-ULTraS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\text{Te}} s_2 \iff s_1 \sim_{\text{Te}, \mathcal{M}^{\text{LC}}_{\mathbb{B} \times \mathbb{B}}} s_2$$ ■

5. Equivalences for Fully Probabilistic Processes

In this section, we extend the work done in the previous section by additionally taking into account the execution probability of transitions. More precisely, we instantiate the three behavioral equivalences of Sect. 3 for fully probabilistic processes represented as functional $\mathbb{R}_{[0,1]}$-ULTraS models (see Sect. 2.3). This is accomplished by introducing a measure function that associates a suitable $\mathbb{R}_{[0,1]}$-value with every triple composed of a source state s, a trace α, and a set of destination states S'. The value $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S')$ computed by this function expresses the probability of performing a computation that is labeled with trace α and leads to a state in S' from state s. Notice that, unlike fully nondeterministic processes where we need two qualitative measure functions for expressing the possibility and the necessity of performing certain computations, here a single quantitative measure function suffices. In this generative setting, we can interpret $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S') > 0$ as the existence of a computation that is labeled with trace α and leads to a state in S' from state s and $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S') = 1$ as the existence of such a computation together with the fact that all the computations from state s of length $|\alpha|$ are labeled with trace α and lead to a state in S'.

Definition 5.1. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{R}_{[0,1]}$-ULTraS. The measure function $\mathcal{M}_{\mathbb{R}_{[0,1]}} : S \times A^* \times 2^S \to \mathbb{R}_{[0,1]}$ for $\mathcal{U}$ is inductively defined in Table 2, where the associated preorder relation is the usual ordering of reals whose minimum is 0. ■

We now show that the three resulting behavioral equivalences $\sim_{\text{B}, \mathcal{M}_{\mathbb{R}_{[0,1]}}}$, $\sim_{\text{Tr}, \mathcal{M}_{\mathbb{R}_{[0,1]}}}$, and $\sim_{\text{Te}, \mathcal{M}^{\text{NPM}}_{\mathbb{R}_{[0,1]}}}$ on functional $\mathbb{R}_{[0,1]}$-ULTraS models – where NPM stands for normalized probability multiplication, a 0-preserving and argument-injective $\mathbb{R}_{[0,1]}$-valued function that we will introduce later on – respectively coincide with the bisimulation, trace, and testing equivalences defined in the literature for GPLTS models.

5.1. Bisimulation Equivalence

Bisimilarity for GPLTS models [61, 79, 125, 120, 8, 43, 117, 2, 29, 74] compares the ability of two fully probabilistic processes of mimicking each other's probabilistic behavior at every step. It is based on the notion of state exit probability. Given a GPLTS $(S, A, \longrightarrow)$, the exit probability of a state $s \in S$ with respect to action $a \in A$ and destination $S' \subseteq S$ is the probability with which s can execute transitions labeled with a that lead to S': $\mathit{prob}_{\text{e}}(s, a, S') = \sum \{\!| p \in \mathbb{R}_{(0,1]} \mid \exists s' \in S'. \, s \xrightarrow{a,p} s' |\!\}$.

Definition 5.2. Let $(S, A, \longrightarrow)$ be a GPLTS. An equivalence relation $\mathcal{B}$ over S is a probabilistic bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$:

$$prob_{\mathrm{e}}(s_1, a, C) \; = \; prob_{\mathrm{e}}(s_2, a, C)$$

We say that $s_1, s_2 \in S$ are probabilistic bisimilar, written $s_1 \sim_{\mathrm{PB}} s_2$, iff there exists a probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

Theorem 5.3. Let $(S, A, \longrightarrow)$ be a GPLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{PB}} s_2 \iff s_1 \sim_{\mathrm{B}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$$ ■

5.2. Trace Equivalence

Trace equivalence for GPLTS models [79, 114, 36, 74] compares the probability with which two fully probabilistic processes perform computations labeled with the same traces. In order to formalize this for a GPLTS $(S, A, \longrightarrow)$, given $s \in S$ we denote by $\mathcal{C}_{\mathrm{fin}}(s)$ the set of finite-length computations of s and by $|c|$ the length of any $c \in \mathcal{C}_{\mathrm{fin}}(s)$. The probability of executing $c \in \mathcal{C}_{\mathrm{fin}}(s)$ is the product of the execution probabilities of the transitions occurring in c:

$$prob(c) \; = \; \begin{cases} 1 & \text{if } |c| = 0 \\ p \cdot prob(c') & \text{if } c \equiv s \xrightarrow{a,p} c' \end{cases}$$

which is lifted to $C \subseteq \mathcal{C}_{\mathrm{fin}}(s)$ as follows:

$$prob(C) \; = \; \sum_{c \in C} prob(c)$$

whenever C is finite and all of its computations are independent of each other, i.e., none of them is a proper prefix of one of the others.

Indicating with $trace(c)$ the sequence of actions labeling the transitions occurring in $c \in \mathcal{C}_{\mathrm{fin}}(s)$, we say that c is compatible with $\alpha \in A^*$ iff $trace(c) = \alpha$ and we denote by $\mathcal{CC}(s, \alpha)$ the set of computations in $\mathcal{C}_{\mathrm{fin}}(s)$ that are compatible with α.

Definition 5.4. Let $(S, A, \longrightarrow)$ be a GPLTS. We say that $s_1, s_2 \in S$ are probabilistic trace equivalent, written $s_1 \sim_{\mathrm{PTr}} s_2$, iff for all traces $\alpha \in A^*$:

$$prob(\mathcal{CC}(s_1, \alpha)) \; = \; prob(\mathcal{CC}(s_2, \alpha))$$ ■

Theorem 5.5. Let $(S, A, \longrightarrow)$ be a GPLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{PTr}} s_2 \iff s_1 \sim_{\mathrm{Tr}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$$ ■

5.3. Testing Equivalence

Testing equivalence for GPLTS models [35, 38, 97, 17, 36] compares the probability with which two fully probabilistic processes pass the same tests. Like in the nondeterministic case, each of the two processes is run in parallel with any test by enforcing synchronization on any action name, with tests being formalized as GPLTS models equipped with a success state.[4] We also extend to interaction systems the notation for the execution probability of computations introduced in Sect. 5.2.

Definition 5.6. A fully probabilistic test is a finite-state, acyclic, and finitely-branching GPLTS $\mathcal{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathcal{T}$ is successful iff its last state is ω. ■

Definition 5.7. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be a GPLTS and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a fully probabilistic test. The interaction system of $\mathcal{L}$ and $\mathcal{T}$ is the GPLTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = (S \times O, A, \longrightarrow)$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.

[4] To be precise, the tests considered in [35] were formalized as deterministic LTS models.

- The transition relation $\longrightarrow \subseteq (S \times O) \times A \times \mathbb{R}_{(0,1]} \times (S \times O)$ is such that $(s, o) \xrightarrow{a,p} (s', o')$ iff $s \xrightarrow{a,p_1}_{\mathcal{L}} s'$ and $o \xrightarrow{a,p_2}_{\mathcal{T}} o'$ with $p = \frac{p_1 \cdot p_2}{q}$, where $q = \sum\{\!| q_1 \cdot q_2 \mid \exists b \in A, s'' \in S, o'' \in O.\, s \xrightarrow{b,q_1}_{\mathcal{L}} s'' \wedge o \xrightarrow{b,q_2}_{\mathcal{T}} o'' |\!\}$ is a normalizing factor. We say that a computation of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ is successful iff its last configuration is successful. Given $s \in S$ and $o \in O$, we denote by $\mathcal{SC}(s, o)$ the set of successful computations in $\mathcal{I}(\mathcal{L}, \mathcal{T})$ from (s, o). ∎

Definition 5.8. Let $(S, A, \longrightarrow)$ be a GPLTS. We say that $s_1, s_2 \in S$ are probabilistic testing equivalent, written $s_1 \sim_{\mathrm{PTe}} s_2$, iff for all every probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$\mathit{prob}(\mathcal{SC}(s_1, o)) \;=\; \mathit{prob}(\mathcal{SC}(s_2, o))$$
∎

Consistent with Def. 5.7, we denote by $\mathrm{NPM} : \mathbb{R}_{[0,1]} \times \mathbb{R}_{[0,1]} \times \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$ the normalized probability multiplication function defined by letting $\mathrm{NPM}(p_1, p_2, q) = \frac{p_1 \cdot p_2}{q}$ when $q > 0$ and $\mathrm{NPM}(p_1, p_2, q) = 0$ when $q = 0$, which is 0-preserving and argument-injective. Given an $\mathbb{R}_{[0,1]}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ and an $\mathbb{R}_{[0,1]}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$, for all configurations (s, o) of their interaction system $\mathcal{I}^{\mathrm{NPM}}(\mathcal{U}, \mathcal{O})$ and for all $a \in A$ we let $(s, o) \xrightarrow{a} \mathcal{D}$ iff $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_1$ and $o \xrightarrow{a}_{\mathcal{O}} \mathcal{D}_2$ with $\mathcal{D}(s', o') = \mathrm{NPM}(\mathcal{D}_1(s'), \mathcal{D}_2(o'), \sum\{\!| \mathcal{D}_1(s'') \cdot \mathcal{D}_2(o'') \mid s'' \in S \wedge o'' \in O |\!\})$ for each $(s', o') \in S \times O$.

In the following, given $s \in S$, $o \in O$, and $\alpha \in A^*$ where S is the set of states of a GPLTS and O is the set of states of a fully probabilistic test, we denote by $\mathcal{SCC}(s, o, \alpha)$ the set of computations in $\mathcal{SC}(s, o)$ that are compatible with α.

Lemma 5.9. Let $(S, A, \longrightarrow)$ be a GPLTS and $s_1, s_2 \in S$. Then $s_1 \sim_{\mathrm{PTe}} s_2$ iff for every fully probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

$$\mathit{prob}(\mathcal{SCC}(s_1, o, \alpha)) \;=\; \mathit{prob}(\mathcal{SCC}(s_2, o, \alpha))$$
∎

Theorem 5.10. Let $(S, A, \longrightarrow)$ be a GPLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{PTe}} s_2 \iff s_1 \sim_{\mathrm{Te}, \mathcal{M}^{\mathrm{NPM}}_{\mathbb{R}_{[0,1]}}} s_2$$
∎

6. Equivalences for Reactive Probabilistic Processes

In this section, we address a different class of probabilistic processes including a limited form of nondeterminism. More precisely, we instantiate the three behavioral equivalences of Sect. 3 for reactive probabilistic processes represented as functional $\mathbb{R}_{[0,1]}$-ULTRAS models (see the first part of Sect. 2.4). This is accomplished by using the same measure function $\mathcal{M}_{\mathbb{R}_{[0,1]}}$ that we have introduced in Def. 5.1 for fully probabilistic processes. In this reactive setting, the value $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S')$ expresses the probability of performing a computation labeled with trace α that leads to a state in S' from state s among all the computations starting at s that are labeled with α. As a consequence, here we can interpret $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S') > 0$ as the existence of a computation that leads to a state in S' from state s among all the computations starting at s that are labeled with α and $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S') = 1$ as the existence of such a computation together with the fact that all the computations from s labeled with α lead to a state in S'.

We now show that the three resulting behavioral equivalences $\sim_{\mathrm{B}, \mathcal{M}_{\mathbb{R}_{[0,1]}}}$, $\sim_{\mathrm{Tr}, \mathcal{M}_{\mathbb{R}_{[0,1]}}}$, and $\sim_{\mathrm{Te}, \mathcal{M}^{\mathrm{PM}}_{\mathbb{R}_{[0,1]}}}$ on functional $\mathbb{R}_{[0,1]}$-ULTRAS models – where the first two have been considered in Sect. 5 under a generative interpretation while in the third one PM stands for probability multiplication, a 0-preserving and argument-injective $\mathbb{R}_{[0,1]}$-valued function that we will introduce later on – respectively coincide with the bisimulation, trace, and testing equivalences defined in the literature for RPLTS models.

6.1. Bisimulation Equivalence

Bisimilarity for RPLTS models [85, 86, 125, 9, 57, 62] is defined in the same way as bisimilarity for GPLTS models (see Def. 5.2), with the difference that in an RPLTS $(S, A, \longrightarrow)$ the exit probability $\mathit{prob}_{\mathrm{e}}(s, a, S')$ of a state $s \in S$ with respect to action $a \in A$ and destination $S' \subseteq S$ has a reactive meaning.

Definition 6.1. Let $(S, A, \longrightarrow)$ be an RPLTS. An equivalence relation $\mathcal{B}$ over S is a probabilistic bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$:

$$prob_{\mathrm{e}}(s_1, a, C) \; = \; prob_{\mathrm{e}}(s_2, a, C)$$

We say that $s_1, s_2 \in S$ are probabilistic bisimilar, written $s_1 \sim_{\mathrm{PB}} s_2$, iff there exists a probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

Theorem 6.2. Let $(S, A, \longrightarrow)$ be an RPLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{PB}} s_2 \iff s_1 \sim_{\mathrm{B}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$$ ■

6.2. Trace Equivalence

Trace equivalence for RPLTS models can be defined in the same way as trace equivalence for GPLTS models (see Def. 5.4), with the difference that in an RPLTS $(S, A, \longrightarrow)$ the execution probability $prob(c)$ of a finite-length computation $c \in \mathcal{C}_{\mathrm{fin}}(s)$ starting from a state $s \in S$ has a reactive meaning. Moreover, given a finite set $C \subseteq \mathcal{C}_{\mathrm{fin}}(s)$ of independent computations, in this reactive setting $prob(C)$ is well defined only if all the computations in C are compatible with the same trace.

To the best of our knowledge, apart from an investigation of algorithmic issues in [122], there is no paper in the literature that extensively deals with trace equivalence for reactive probabilistic processes. The closest work is the second part of [114], where a trace-based relation is defined for processes in which the choice among transitions labeled with the same action is probabilistic, whereas the choice among transitions labeled with different actions can be probabilistic or nondeterministic.

Definition 6.3. Let $(S, A, \longrightarrow)$ be an RPLTS. We say that $s_1, s_2 \in S$ are probabilistic trace equivalent, written $s_1 \sim_{\mathrm{PTr}} s_2$, iff for all traces $\alpha \in A^*$:

$$prob(\mathcal{CC}(s_1, \alpha)) \; = \; prob(\mathcal{CC}(s_2, \alpha))$$ ■

Theorem 6.4. Let $(S, A, \longrightarrow)$ be an RPLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{PTr}} s_2 \iff s_1 \sim_{\mathrm{Tr}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$$ ■

6.3. Testing Equivalence

Testing equivalence for RPLTS models [83] compares the probability with which two reactive probabilistic processes pass the same tests. Like in the fully probabilistic case, each of the two processes is run in parallel with any test by enforcing synchronization on any action name, with tests being formalized as RPLTS models equipped with a success state.[5] Unlike the fully probabilistic case, due to the presence of a limited form of nondeterminism, there is not necessarily a single probability value with which a process passes a test. In general, there can be several values, each of which depends on how at each step the nondeterministic choice is solved among transitions labeled with different actions enabled both in the process and in the test. When considering two reactive probabilistic processes and a test, it is thus necessary to compute for every trace of the interaction system of each process the probability of performing a successful computation compatible with that trace. Then, one option is to compare for the two processes the suprema ($\bigsqcup$) and the infima ($\bigsqcap$) of these values over all traces of the two interaction systems.

Given a reactive probabilistic process and a test, taking the supremum and the infimum of the values mentioned above is a natural extension of testing equivalence for fully nondeterministic processes: when the supremum is greater than zero then it means that the reactive probabilistic process may pass the test, whereas when the infimum is equal to one then it means that the process must pass the test. From a different perspective, we note that with every maximal computation of the interaction system of a fully nondeterministic process and a test we could associate a truth value indicating whether that computation is successful or not. Then, in the may-testing case we should consider the supremum of these truth values

[5] To be precise, the tests considered in [83] were formalized as possibly replicated deterministic LTS models.

(i.e., their logical disjunction), whilst in the must-testing case we should consider the infimum of these truth values (i.e., their logical conjunction). When dealing with a reactive probabilistic process, the difference is that we associate a probability value rather than a truth value. This is possible only after solving all nondeterministic choices, with each resolution of nondeterminism corresponding to the selection of a specific trace labeling a maximal computation of the interaction system.

Definition 6.5. A reactive probabilistic test is a finite-state, acyclic, and finitely-branching RPLTS $\mathcal{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathcal{T}$ is successful iff its last state is ω. ■

Definition 6.6. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be an RPLTS and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a reactive probabilistic test. The interaction system of $\mathcal{L}$ and $\mathcal{T}$ is the RPLTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = (S \times O, A, \longrightarrow)$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.
- The transition relation $\longrightarrow \subseteq (S \times O) \times A \times \mathbb{R}_{(0,1]} \times (S \times O)$ is such that $(s, o) \xrightarrow{a,p} (s', o')$ iff $s \xrightarrow{a,p_1}_{\mathcal{L}} s'$ and $o \xrightarrow{a,p_2}_{\mathcal{T}} o'$ with $p = p_1 \cdot p_2$. We say that a computation of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ is successful iff its last configuration is successful. Given $s \in S$, $o \in O$, and $\alpha \in A^*$, we denote by $\mathcal{SCC}(s, o, \alpha)$ the set of successful computations in $\mathcal{I}(\mathcal{L}, \mathcal{T})$ from (s, o) that are compatible with α. ■

In the following, given $s \in S$ and $o \in O$ where S is the set of states of an RPLTS and O is the set of states of a reactive probabilistic test, we denote by $Tr_{\max}(s, o)$ the set of traces labeling the maximal computations from (s, o). We will not consider traces in $A^* \setminus Tr_{\max}(s, o)$ when computing the supremum and the infimum of the probabilities of performing a successful computation compatible with some trace, because otherwise for the infimum we would always get 0.

Definition 6.7. Let $(S, A, \longrightarrow)$ be an RPLTS. We say that $s_1, s_2 \in S$ are probabilistic testing equivalent, written $s_1 \sim_{\mathrm{PTe}} s_2$, iff for every reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$\bigsqcup_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) = \bigsqcup_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha))$$
$$\bigsqcap_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) = \bigsqcap_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha))$$
■

Consistent with Def. 6.6, we denote by $\mathrm{PM} : \mathbb{R}_{[0,1]} \times \mathbb{R}_{[0,1]} \to \mathbb{R}_{[0,1]}$ the probability multiplication function defined by letting $\mathrm{PM}(p_1, p_2) = p_1 \cdot p_2$, which is 0-preserving and argument-injective. Given an $\mathbb{R}_{[0,1]}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ and an $\mathbb{R}_{[0,1]}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$, for all configurations (s, o) of their interaction system $\mathcal{I}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})$ and for all $a \in A$ we let $(s, o) \xrightarrow{a} \mathcal{D}$ iff $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_1$ and $o \xrightarrow{a}_{\mathcal{O}} \mathcal{D}_2$ with $\mathcal{D}(s', o') = \mathrm{PM}(\mathcal{D}_1(s'), \mathcal{D}_2(o'))$ for each $(s', o') \in S \times O$.

Lemma 6.8. Let $(S, A, \longrightarrow)$ be an RPLTS and $s_1, s_2 \in S$. Then $s_1 \sim_{\mathrm{PTe}} s_2$ iff for every reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

$$prob(\mathcal{SCC}(s_1, o, \alpha)) = prob(\mathcal{SCC}(s_2, o, \alpha))$$
■

Theorem 6.9. Let $(S, A, \longrightarrow)$ be an RPLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{PTe}} s_2 \iff s_1 \sim_{\mathrm{Te}, \mathcal{M}^{\mathrm{PM}}_{\mathbb{R}_{[0,1]}}} s_2$$
■

It is worth pointing out that, due to Lemmas 5.9 and 6.8 coming from [25], probabilistic testing equivalence for fully probabilistic processes and probabilistic testing equivalence for reactive probabilistic processes could have been defined in the same way in the literature, as was the case with probabilistic bisimulation equivalence and probabilistic trace equivalence for the same two classes of probabilistic processes.

$$\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}(s,\alpha,S') \; = \; \begin{cases} \bigcup\limits_{s \xrightarrow{a} \mathcal{D}} \{ \sum\limits_{s' \in S} \mathcal{D}(s') \cdot p_{s'} \mid p_{s'} \in \mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}(s',\alpha',S') \} & \\ & \text{if } \alpha = a \circ \alpha' \text{ and } \exists \mathcal{D} \in [S \rightarrow \mathbb{R}_{[0,1]}].\, s \xrightarrow{a} \mathcal{D} \\ \{1\} & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ \{0\} & \text{if } \alpha = a \circ \alpha' \text{ and } \nexists \mathcal{D} \in [S \rightarrow \mathbb{R}_{[0,1]}].\, s \xrightarrow{a} \mathcal{D} \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$$

Table 3: Measure function for $\mathbb{R}_{[0,1]}$-ULTRAS models representing nondeterministic and probabilistic processes

7. Equivalences for Nondeterministic and Probabilistic Processes

In this section, we examine another class of probabilistic processes including internal nondeterminism. More precisely, we instantiate the three behavioral equivalences of Sect. 3 for nondeterministic and probabilistic processes represented as $\mathbb{R}_{[0,1]}$-ULTRAS models (see the second part of Sect. 2.4). This is accomplished by introducing a measure function that associates a suitable *nonempty set* of $\mathbb{R}_{[0,1]}$-values with every triple composed of a source state s, a trace α, and a set of destination states S'. Let us denote by $2^{\mathbb{R}_{[0,1]}}_{\cdot}$ the set of nonempty subsets of $\mathbb{R}_{[0,1]}$. The set $\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}(s,\alpha,S')$ computed by the previously mentioned function contains for each possible way of resolving nondeterminism the probability of performing a computation that is labeled with trace α and leads to a state in S' from state s. The set $\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}(s,\alpha,S')$ boils down to $\{\mathcal{M}_{\mathbb{R}_{[0,1]}}(s,\alpha,S')\}$ when there is no internal nondeterminism, i.e., when for every state of the considered $\mathbb{R}_{[0,1]}$-ULTRAS the actions labeling the outgoing transitions are all different from each other.

Definition 7.1. Let $\mathcal{U} = (S, A, \longrightarrow)$ be an $\mathbb{R}_{[0,1]}$-ULTRAS. The measure function $\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}} : S \times A^* \times 2^S \rightarrow 2^{\mathbb{R}_{[0,1]}}_{\cdot}$ for $\mathcal{U}$ is inductively defined in Table 3, where the associated preorder relation, whose minimum is $\{0\}$, is defined by letting $R_1 \sqsubseteq_{2^{\mathbb{R}_{[0,1]}}_{\cdot}} R_2$ iff $\bigsqcap R_1 \leq \bigsqcap R_2$ and $|R_1| \leq |R_2|$.[6] ■

Unlike all previous cases, we now show that the three resulting behavioral equivalences $\sim_{\mathrm{B},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}}$, $\sim_{\mathrm{Tr},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}}$, and $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{PM}}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}}$ on $\mathbb{R}_{[0,1]}$-ULTRAS models – where PM is the probability multiplication function introduced in Sect. 6.3 – do *not* coincide with the bisimulation, trace, and testing equivalences defined in the literature for NPLTS models. They are shown to coincide instead with new behavioral equivalences for NPLTS models recently studied in [26, 25].

7.1. Bisimulation Equivalence

Bisimilarity for NPLTS models [65, 112, 13, 92, 113, 51, 70, 67, 9, 32, 41] combines bisimilarity for fully nondeterministic processes with bisimilarity for fully or reactive probabilistic processes, with the former extended to state distributions and the latter abstracting from action names.[7] Given an NPLTS $(S, A, \longrightarrow)$, $\mathcal{D} \in [S \rightarrow \mathbb{R}_{[0,1]}]$, and $S' \subseteq S$, in the following we let $\mathcal{D}(S') = \sum_{s' \in S'} \mathcal{D}(s')$.

Definition 7.2. Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a probabilistic class-distribution bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ with $\mathcal{D}_1(C) = \mathcal{D}_2(C)$ for all equivalence classes $C \in S/\mathcal{B}$. We say that $s_1, s_2 \in S$ are probabilistic class-distribution bisimilar, written $s_1 \sim_{\mathrm{PB,dis}} s_2$, iff there exists a probabilistic class-distribution bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

[6] The constraint on set cardinalities ensures the existence of the minimum element $\{0\}$. Without this constraint, there would be infinitely many minimal elements, which are all the sets containing 0, but no minimum element.

[7] In [112], also a coarser bisimilarity for NPLTS models was defined, which additionally allows convex combinations of equally labeled transitions to be considered in the bisimulation game.

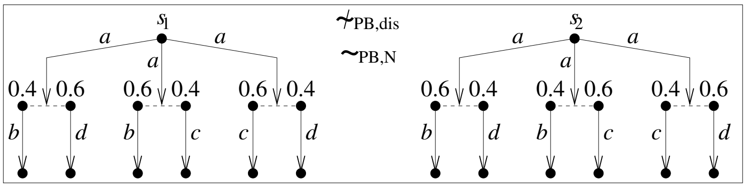

Figure 2: Counterexample showing that $\sim_{\text{PB,dis}}$ is strictly finer than $\sim_{\text{PB,N}}$

The bisimulation equivalence $\sim_{\text{B},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}}$ does not capture the bisimulation equivalence $\sim_{\text{PB,dis}}$ of Def. 7.2. The reason is that the latter matches transitions on the basis of *class distributions*, which means that for each transition of one of two bisimilar states there must exist an equally labeled transition of the other state such that, *for every equivalence class*, the two transitions have the same probability of reaching a state in that class. Similar to [45, 121], this constraint can be relaxed by considering *a single equivalence class at a time*, i.e., by anticipating the quantification over equivalence classes. In this way, a transition departing from one of the two states can be matched, with respect to the probabilities associated with different classes, by several different transitions departing from the other state, which leads to the following coarser bisimulation equivalence for NPLTS models recently studied in [26] that is captured by $\sim_{\text{B},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}}$. Groups of equivalence classes are considered so that, similar to [116] and unlike $\sim_{\text{PB,dis}}$, the new bisimulation equivalence is characterized by a minor variant of a standard probabilistic logic, which is PML [85]. We have also that a slight variant of the new bisimulation equivalence in which $\leq$ is used in place of $=$ is precisely characterized by PML. Figure 2 illustrates the difference between $\sim_{\text{PB,dis}}$ and the new equivalence.

Definition 7.3. Let $(S, A, \longrightarrow)$ be an NPLTS. An equivalence relation $\mathcal{B}$ over S is a probabilistic bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ with $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$. We say that $s_1, s_2 \in S$ are probabilistic bisimilar, written $s_1 \sim_{\text{PB,N}} s_2$, iff there exists a probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

Theorem 7.4. Let $(S, A, \longrightarrow)$ be an NPLTS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\text{PB,N}} s_2 \iff s_1 \sim_{\text{B},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}} s_2$$

■

7.2. Trace Equivalence

Trace equivalence for NPLTS models [110, 44, 34, 88, 101] combines trace equivalence for fully nondeterministic processes with trace equivalence for fully or reactive probabilistic processes, where the probability of performing computations labeled with the same traces are examined for each possible way of resolving nondeterminism. In order to formalize this for an NPLTS $\mathcal{L}$, given a state s of $\mathcal{L}$ we take the set of resolutions of s. Each of them is a tree-like structure whose branching points represent probabilistic choices. This is obtained by unfolding from s the graph structure underlying $\mathcal{L}$ and by selecting at each state – without considering how that state has been reached – a single transition of $\mathcal{L}$ – (memoryless) deterministic scheduler – or a convex combination of equally labeled transitions of $\mathcal{L}$ – (memoryless) randomized scheduler – among all the transitions possible from that state. Every resolution of s corresponds to a computation in $\mathcal{L}$ from s whenever each transition of $\mathcal{L}$ concentrates all the probability mass into a single target state. Below, we restrict ourselves to resolutions arising from deterministic schedulers, described as NPLTS models in which every state has at most one outgoing transition so that nondeterminism is completely absent.[8]

Definition 7.5. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be an NPLTS and $s \in S$. We say that an NPLTS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ is a resolution of s obtained via a deterministic scheduler iff there exists a state correspondence function $corr : Z \to S$ such that $s = corr(z_s)$, for some $z_s \in Z$, and for all $z \in Z$:

[8]In [110, 88, 101], also randomized schedulers are admitted thus ending up with a coarser trace equivalence in which convex combinations of equally labeled transitions are taken into account.

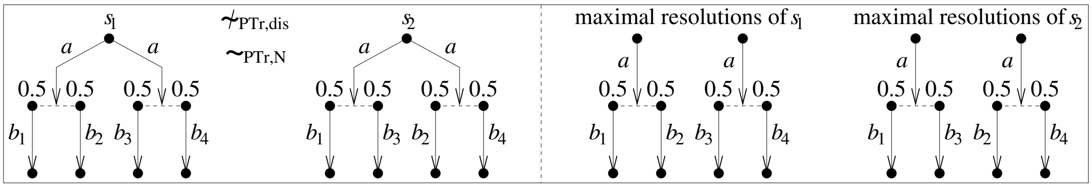

Figure 3: Counterexample showing that $\sim_{\text{PTr,dis}}$ is strictly finer than $\sim_{\text{PTr,N}}$

- If $z \xrightarrow{a}_{\mathcal{Z}} \mathcal{D}$, then $corr(z) \xrightarrow{a}_{\mathcal{L}} \mathcal{D}'$ with $\mathcal{D}(z') = \mathcal{D}'(corr(z'))$ for all $z' \in Z$.
- If $z \xrightarrow{a_1}_{\mathcal{Z}} \mathcal{D}_1$ and $z \xrightarrow{a_2}_{\mathcal{Z}} \mathcal{D}_2$, then $a_1 = a_2$ and $\mathcal{D}_1 = \mathcal{D}_2$. ■

Given a state s of an NPLTS $\mathcal{L}$, we denote by $Res(s)$ the set of resolutions of s obtained via deterministic schedulers. Given $\mathcal{Z} \in Res(s)$ and $c \in \mathcal{C}_{\text{fin}}(z_s)$, due to the absence of nondeterminism in $\mathcal{Z}$ the probability of executing c can be defined as in Sect. 5.2 – with $prob(c)$ being always equal to 1 if each transition of $\mathcal{L}$ concentrates all the probability mass into a single target state – and can be lifted as usual to any $C \subseteq \mathcal{C}_{\text{fin}}(z_s)$ such that C is finite and all of its computations are independent of each other. In the following, we denote by $\mathcal{CC}(z_s, \alpha)$ the set of computations in $\mathcal{C}_{\text{fin}}(z_s)$ that are compatible with $\alpha \in A^*$.

Definition 7.6. Let $(S, A, \longrightarrow)$ be an NPLTS. We say that $s_1, s_2 \in S$ are probabilistic trace-distribution equivalent, written $s_1 \sim_{\text{PTr,dis}} s_2$, iff:

- For each resolution $\mathcal{Z}_1 \in Res(s_1)$ there exists a resolution $\mathcal{Z}_2 \in Res(s_2)$ such that for all traces $\alpha \in A^*$:
$$prob(\mathcal{CC}(z_{s_1}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_2}, \alpha))$$
- For each resolution $\mathcal{Z}_2 \in Res(s_2)$ there exists a resolution $\mathcal{Z}_1 \in Res(s_1)$ such that for all traces $\alpha \in A^*$:
$$prob(\mathcal{CC}(z_{s_2}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_1}, \alpha))$$ ■

The trace equivalence $\sim_{\text{Tr},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}}$ does not capture the trace equivalence $\sim_{\text{PTr,dis}}$ of Def. 7.6. The reason is that the latter matches resolutions on the basis of *trace distributions*, which means that for each resolution of one of two trace equivalent states there must exist a resolution of the other state such that, *for every trace*, the two resolutions have the same probability of performing a computation labeled with that trace. This constraint can be relaxed by considering *a single trace at a time*, i.e., by anticipating the quantification over traces. In this way, differently labeled computations of a resolution of one of the two states are allowed to be matched by computations of several different resolutions of the other state, which leads to the following coarser trace equivalence for NPLTS models captured by $\sim_{\text{Tr},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}}$ that has been recently shown to be compositional in the full version of [25]. Figure 3 illustrates the difference between the two equivalences.

Definition 7.7. Let $(S, A, \longrightarrow)$ be an NPLTS. We say that $s_1, s_2 \in S$ are probabilistic trace equivalent, written $s_1 \sim_{\text{PTr,N}} s_2$, iff for all traces $\alpha \in A^*$:

- For each resolution $\mathcal{Z}_1 \in Res(s_1)$ there exists a resolution $\mathcal{Z}_2 \in Res(s_2)$ such that:
$$prob(\mathcal{CC}(z_{s_1}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_2}, \alpha))$$
- For each resolution $\mathcal{Z}_2 \in Res(s_2)$ there exists a resolution $\mathcal{Z}_1 \in Res(s_1)$ such that:
$$prob(\mathcal{CC}(z_{s_2}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_1}, \alpha))$$ ■

Theorem 7.8. Let $(S, A, \longrightarrow)$ be an NPLTS. For all $s_1, s_2 \in S$:
$$s_1 \sim_{\text{PTr,N}} s_2 \iff s_1 \sim_{\text{Tr},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}} s_2$$ ■

7.3. Testing Equivalence

Testing equivalence for NPLTS models [129, 76, 77, 111, 94, 78, 33, 99, 54, 52, 53] is an extension of testing equivalence for reactive probabilistic processes. Like in all the previous cases, each of the two processes is run in parallel with any test by enforcing synchronization on any action name, with tests being formalized as NPLTS models equipped with a success state. Like in the reactive probabilistic case, there is not a single probability value with which a process passes a test. Unlike the reactive probabilistic case, due to the possible presence of equally labeled transitions departing from the same state, it is not enough to consider for each trace the probability of performing a successful computation compatible with that trace, i.e., traces are no longer sufficient to resolve nondeterminism. Given two nondeterministic and probabilistic processes and a test, we thus need to compute the probability of performing a successful computation *in every resolution* of the interaction system of each process and the test. Then, like in the reactive probabilistic case, a natural option is to compare for the two processes the suprema and the infima of these values over all resolutions of the two interaction systems.[9]

Definition 7.9. A nondeterministic and probabilistic test is a finite-state, acyclic, and finitely-branching NPLTS $\mathcal{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathcal{T}$ is successful iff its last state is ω. ■

Definition 7.10. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be an NPLTS and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a nondeterministic and probabilistic test. The interaction system of $\mathcal{L}$ and $\mathcal{T}$ is the NPLTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = (S \times O, A, \longrightarrow)$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.
- The transition relation $\longrightarrow \subseteq (S \times O) \times A \times [(S \times O) \to \mathbb{R}_{[0,1]}]$ is such that $(s, o) \xrightarrow{a} \mathcal{D}$ iff $s \xrightarrow{a}_{\mathcal{L}} \mathcal{D}_1$ and $o \xrightarrow{a}_{\mathcal{T}} \mathcal{D}_2$ with $\mathcal{D}(s', o') = PM(\mathcal{D}_1(s'), \mathcal{D}_2(o')) = \mathcal{D}_1(s') \cdot \mathcal{D}_2(o')$ for each $(s', o') \in S \times O$. We say that a computation of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ is successful iff its last configuration is successful. Given $s \in S$, $o \in O$, and $\mathcal{Z} \in Res(s, o)$, we denote by $\mathcal{SC}(z_{s,o})$ the set of successful computations in $\mathcal{Z}$ from $z_{s,o}$. ■

In the following, given $s \in S$ and $o \in O$ where S is the set of states of an NPLTS and O is the set of states of a nondeterministic and probabilistic test, we denote by $Res_{\max}(s, o)$ the set of resolutions in $Res(s, o)$ that are maximal, i.e., that cannot be further extended in accordance with the graph structure of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ and the constraints of Def. 7.5. We will not consider resolutions in $Res(s, o) \setminus Res_{\max}(s, o)$ when computing the supremum and the infimum of the probabilities of performing a successful computation, because otherwise for the infimum we would always get 0.

Definition 7.11. Let $(S, A, \longrightarrow)$ be an NPLTS. We say that $s_1, s_2 \in S$ are probabilistic $\sqcup\sqcap$-testing equivalent, written $s_1 \sim_{\mathrm{PTe},\sqcup\sqcap} s_2$, iff for every nondeterministic and probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$\bigsqcup_{\mathcal{Z}_1 \in Res_{\max}(s_1, o)} prob(\mathcal{SC}(z_{s_1,o})) = \bigsqcup_{\mathcal{Z}_2 \in Res_{\max}(s_2, o)} prob(\mathcal{SC}(z_{s_2,o}))$$

$$\bigsqcap_{\mathcal{Z}_1 \in Res_{\max}(s_1, o)} prob(\mathcal{SC}(z_{s_1,o})) = \bigsqcap_{\mathcal{Z}_2 \in Res_{\max}(s_2, o)} prob(\mathcal{SC}(z_{s_2,o}))$$

■

The testing equivalence $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{PM}}_{2^{\mathbb{R}_{[0,1]}}}}$ does not capture the testing equivalence $\sim_{\mathrm{PTe},\sqcup\sqcap}$ of Def. 7.11. The reason is that the former is based on a measure function that computes for every trace the *set of probabilities* of performing a successful computation compatible with that trace corresponding to all possible ways of resolving nondeterminism in the interaction system, whereas the latter considers only the *maximal and minimal overall probability* of performing a successful computation, with overall probability meaning the sum over all traces of the probability of performing a successful computation compatible with a trace in a certain resolution.

[9] In [111], also resolutions coming from randomized schedulers are admitted thus ending up with a coarser testing equivalence in which convex combinations of equally labeled transitions are taken into account.

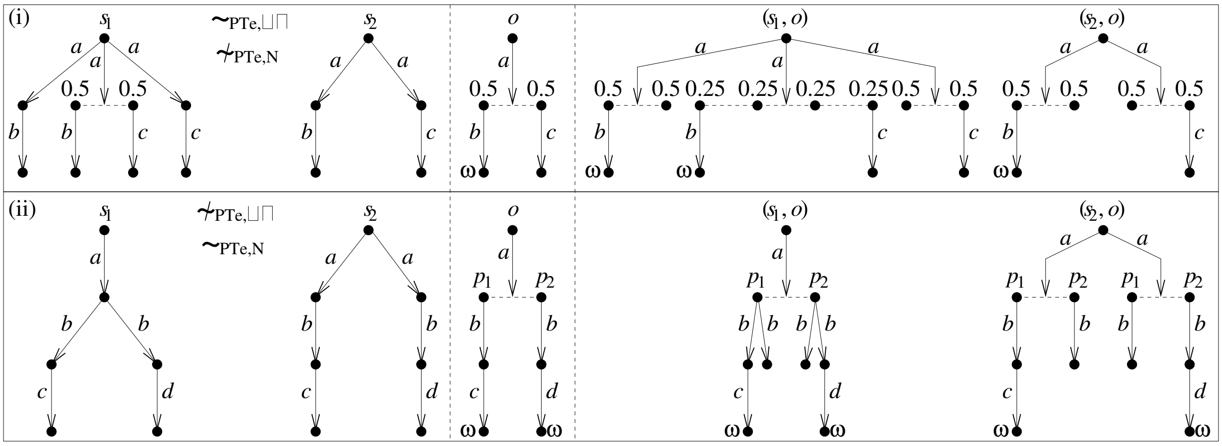

Figure 4: Counterexamples showing that $\sim_{\mathrm{PTe},\sqcup\sqcap}$ and $\sim_{\mathrm{PTe,N}}$ are incomparable

The testing equivalence $\sim_{\mathrm{PTe},\sqcup\sqcap}$ of Def. 7.11 is known to suffer from a couple of anomalies. As noted in [25], it is included in neither of the two trace equivalences of Defs. 7.6 and 7.7 introduced for the same class of processes.[10] Most importantly, it has an alternative characterization based on a simulation-like equivalence rather than a trace-like equivalence, thus discriminating more than classical testing equivalence over fully nondeterministic processes [46]. In fact, $\sim_{\mathrm{PTe},\sqcup\sqcap}$ differentiates processes that perform the same sequence of actions with the same probability but make internal choices in different moments and thus, when applied to processes without probabilities, does not coincide with $\sim_{\mathrm{Te}}$. This is a consequence of the presence of probabilistic choices within tests, as they make it possible to take copies of the intermediate states of the processes under test thereby enhancing the discriminating power of testing equivalence [1]. On the probabilistic side, this results in an unrealistic estimation of success probabilities [60].

The previously mentioned anomalies of $\sim_{\mathrm{PTe},\sqcup\sqcap}$ can be overcome through the following testing equivalence for NPLTS models recently studied in [25] that is captured by $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{PM}}_{2^{\mathbb{R}_{[0,1]}}}}$. To counterbalance the strong discriminating power deriving from the presence of probabilistic choices within tests, the idea is to match the resolutions of the two interaction systems on the basis of *all of their success probabilities* where – building on the results of Lemmas 5.9 and 6.8 – such probabilities are considered in a *trace-by-trace fashion* instead of being overall probabilities. Figure 4 illustrates the difference between the two equivalences by showing two distinguishing tests. In the following, given $s \in S$, $o \in O$, $\mathcal{Z} \in \mathit{Res}(s, o)$, and $\alpha \in A^*$ where S is the set of states of an NPLTS and O is the set of states of a nondeterministic and probabilistic test, we denote by $\mathcal{SCC}(z_{s,o}, \alpha)$ the set of computations in $\mathcal{SC}(z_{s,o})$ that are compatible with α.

Definition 7.12. Let $(S, A, \longrightarrow)$ be an NPLTS. We say that $s_1, s_2 \in S$ are probabilistic testing equivalent, written $s_1 \sim_{\mathrm{PTe,N}} s_2$, iff for every nondeterministic and probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all traces $\alpha \in A^*$:

- For each resolution $\mathcal{Z}_1 \in \mathit{Res}(s_1, o)$ there exists a resolution $\mathcal{Z}_2 \in \mathit{Res}(s_2, o)$ such that:
$$\mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha)) \; = \; \mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
- For each resolution $\mathcal{Z}_2 \in \mathit{Res}(s_2, o)$ there exists a resolution $\mathcal{Z}_1 \in \mathit{Res}(s_1, o)$ such that:
$$\mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha)) \; = \; \mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha))$$
■

Theorem 7.13. Let $(S, A, \longrightarrow)$ be an NPLTS. For all $s_1, s_2 \in S$:
$$s_1 \sim_{\mathrm{PTe,N}} s_2 \iff s_1 \sim_{\mathrm{Te},\mathcal{M}^{\mathrm{PM}}_{2^{\mathbb{R}_{[0,1]}}}} s_2$$
■

[10] Inclusion would hold if randomized schedulers were admitted as in [111].

$\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t) \;=\; \begin{cases} \int_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \mathcal{M}_{\mathrm{ete}}(s', \alpha', S')(t - x)\,\mathrm{d}x & \text{if } \alpha = a \circ \alpha' \text{ and } \mathrm{E}(s) > 0 \\ 1 & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ 0 & \text{if } \alpha \neq \varepsilon \text{ and } \mathrm{E}(s) = 0 \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$	
$\mathcal{M}_{\mathrm{sbs}}(s, \alpha, S')(\theta) \;=\; \begin{cases} (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot \sum_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \mathcal{M}_{\mathrm{sbs}}(s', \alpha', S')(\theta') & \text{if } \alpha = a \circ \alpha' \text{ and } \theta = t \circ \theta' \text{ and } \mathrm{E}(s) > 0 \\ 1 & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ 0 & \text{if } \alpha \neq \varepsilon \text{ and } \theta = \varepsilon \\ & \text{or } \alpha \neq \varepsilon \text{ and } \theta \neq \varepsilon \text{ and } \mathrm{E}(s) = 0 \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$	

Table 4: Measure functions for functional $\mathbb{R}_{\geq 0}$-ULTRAS models representing generative stochastic processes

8. Equivalences for Fully Stochastic Processes

In this section, we further extend the work done in the previous sections by additionally taking into account a notion of time formalized by means of exponentially distributed random variables that quantify the durations of transitions. More precisely, we instantiate the three behavioral equivalences of Sect. 3 for fully stochastic processes – involving only exponential distributions – represented as functional $\mathbb{R}_{\geq 0}$-ULTRAS models (see Sect. 2.3). Given one such process, we remind that $\mathrm{E}(s)$ denotes the sum of the rates of the transitions departing from state s and, when $\mathrm{E}(s) > 0$, it holds that:

- Function $\mathrm{Exp}_s(t) = 1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}$ is the probability of leaving state s within time t, which is the cumulative distribution function of an exponentially distributed random variable with rate $\mathrm{E}(s)$.
- Value $\frac{1}{\mathrm{E}(s)}$ is the average sojourn time in state s, which is the expected value of an exponentially distributed random variable with rate $\mathrm{E}(s)$.
- Value $\frac{\lambda}{\mathrm{E}(s)}$ is the probability of executing a transition departing from s whose rate is $\lambda \in \mathbb{R}_{>0}$.

Unlike the previous sections, when defining the measure function we distinguish between two cases. The measure function for the *end-to-end* case associates a suitable $\mathbb{R}_{[0,1]}$-valued *function* with every triple composed of a source state s, a trace α, and a set of destination states S', which is parameterized with respect to the end-to-end delay $t \in \mathbb{R}_{\geq 0}$ of the trace. The value $\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t)$ computed by this function expresses the probability of performing within time t a computation that is labeled with trace α and leads to a state in S' from state s. The subscript "ete" is a symbolic shorthand for $[\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{[0,1]}]$.

Definition 8.1. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\geq 0}$-ULTRAS. The end-to-end measure function $\mathcal{M}_{\mathrm{ete}} : S \times A^* \times 2^S \rightarrow [\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{[0,1]}]$ for $\mathcal{U}$ is inductively defined in the first part of Table 4, where the associated preorder relation, whose minimum is the function mapping every element of $\mathbb{R}_{\geq 0}$ to 0, is defined by letting $f_1 \sqsubseteq_{\mathrm{ete}} f_2$ iff $f_1(t) \leq f_2(t)$ for all $t \in \mathbb{R}_{\geq 0}$. ■

The value $\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t)$ is the probability of the set of computations that are labeled with trace α and lead to a state in S' from state s within t time units. If there are no such computations, then $\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t) = 0$, otherwise $\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t) \in \mathbb{R}_{(0,1]}$. In the case $\alpha = a \circ \alpha'$ and $\mathrm{E}(s) > 0$, this value is computed as the convolution of two probability distributions. Assuming to spend $x \in \mathbb{R}_{[0,t]}$ time units in state s, the first operand of the convolution is the exponentially distributed density function quantifying the sojourn time in s, i.e., the derivative of $\mathrm{Exp}_s(t)$ evaluated in x. For each state s' reachable from s by executing a, the first operand is multiplied by the probability of the set of computations that are labeled with the remaining trace α' and lead to a state in S' from state s' within the remaining $t - x$ time units.

The measure function for the *step-by-step* case associates a suitable $\mathbb{R}_{[0,1]}$-valued *function* with every triple composed of a source state s, a trace α, and a set of destination states S', which is parameterized with respect to the step-by-step delay $\theta \in (\mathbb{R}_{\geq 0})^*$ of the trace. The value $\mathcal{M}_{\mathrm{sbs}}(s, \alpha, S')(\theta)$ computed by this function expresses the probability of performing within time θ a computation that is labeled with trace α and leads to a state in S' from state s. The subscript "sbs" is a symbolic shorthand for $[(\mathbb{R}_{\geq 0})^* \rightarrow \mathbb{R}_{[0,1]}]$.

Definition 8.2. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\geq 0}$-ULTRAS. The step-by-step measure function $\mathcal{M}_{\mathrm{sbs}} : S \times A^* \times 2^S \rightarrow [(\mathbb{R}_{\geq 0})^* \rightarrow \mathbb{R}_{[0,1]}]$ for $\mathcal{U}$ is inductively defined in the second part of Table 4, where the associated preorder relation, whose minimum is the function mapping every element of $(\mathbb{R}_{\geq 0})^*$ to 0, is defined by letting $f_1 \sqsubseteq_{\mathrm{sbs}} f_2$ iff $f_1(\theta) \leq f_2(\theta)$ for all $\theta \in (\mathbb{R}_{\geq 0})^*$. ■

The value $\mathcal{M}_{\mathrm{sbs}}(s, \alpha, S')(\theta)$ is the probability of the set of computations that are labeled with trace α and lead to a state in S' from state s, such that the average sojourn time in the i-th state traversed by any such computation is not greater than $\theta[i]$ for each i ranging from 1 to the length of the computation. If there are no such computations, then $\mathcal{M}_{\mathrm{sbs}}(s, \alpha, S')(\theta) = 0$, otherwise $\mathcal{M}_{\mathrm{sbs}}(s, \alpha, S')(\theta) \in \mathbb{R}_{(0,1]}$. In the case $\alpha = a \circ \alpha'$ and $\theta = t \circ \theta'$ and $\mathrm{E}(s) > 0$, this value is computed as the product of two probability distributions. The first operand of the product is the probability of leaving state s within t time units, i.e., $\mathrm{Exp}_s(t)$. For each state s' reachable from s by executing a, the first operand is multiplied by the probability of the set of computations that are labeled with the remaining trace α' and lead to a state in S' from state s' within the remaining sequence θ' of time units.

We now show that the six resulting behavioral equivalences $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}$, $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}}$, $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete}}}$, $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs}}}$, $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete}}}$, and $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs}}}$ on functional $\mathbb{R}_{\geq 0}$-ULTRAS models – where RM stands for rate multiplication, a 0-preserving and argument-injective $\mathbb{R}_{\geq 0}$-valued function that we will introduce later on – respectively coincide with the end-to-end and step-by-step bisimulation, trace, and testing equivalences defined in the literature for GMLTS models. In particular, we will see that $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}$ and $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}}$ coincide, whereas the end-to-end trace and testing equivalences are respectively different from the step-by-step trace and testing equivalences.

8.1. Bisimulation Equivalence

Bisimilarity for GMLTS models [73, 71, 37, 21, 55] compares the ability of two fully stochastic processes of mimicking each other's stochastic behavior at every step. It is based on the notion of state exit rate. Given a GMLTS $(S, A, \longrightarrow)$, the exit rate of a state $s \in S$ with respect to action $a \in A$ and destination $S' \subseteq S$ is the rate at which s can execute transitions labeled with a that lead to S', which is the sum of the rates of those transitions due to the fact that transition firing is governed by the race policy: $\mathit{rate}_{\mathrm{e}}(s, a, S') = \sum\{\!| \lambda \in \mathbb{R}_{>0} \mid \exists s' \in S'.\, s \xrightarrow{a,\lambda} s' |\!\}$.

Definition 8.3. Let $(S, A, \longrightarrow)$ be a GMLTS. An equivalence relation $\mathcal{B}$ over S is a Markovian bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$:

$$\mathit{rate}_{\mathrm{e}}(s_1, a, C) \; = \; \mathit{rate}_{\mathrm{e}}(s_2, a, C)$$

We say that $s_1, s_2 \in S$ are Markovian bisimilar, written $s_1 \sim_{\mathrm{MB}} s_2$, iff there exists a Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

If we define the *rate-based* exit probability by letting $\mathit{prob}_{\mathrm{e}}(s, a, S') = \mathit{rate}_{\mathrm{e}}(s, a, S')/\mathrm{E}(s)$ when $\mathrm{E}(s) > 0$ and $\mathit{prob}_{\mathrm{e}}(s, a, S') = 0$ when $\mathrm{E}(s) = 0$, the definition above is equivalent to requiring that, whenever $(s_1, s_2) \in \mathcal{B}$, then $\mathrm{E}(s_1) = \mathrm{E}(s_2)$ and for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$:

$$\mathit{prob}_{\mathrm{e}}(s_1, a, C) \; = \; \mathit{prob}_{\mathrm{e}}(s_2, a, C)$$

Note that condition $\mathrm{E}(s_1) = \mathrm{E}(s_2)$, which takes into account only time-related information, represents the difference between bisimilarity for GMLTS models and bisimilarity for GPLTS models (see Sect. 5.1).

We prove that $\sim_{\mathrm{MB}}$ coincides with $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}$ and $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}}$, which implies that for fully stochastic processes there is no difference between the end-to-end case and the step-by-step case in the bisimulation approach.

Lemma 8.4. Let $(S, A, \longrightarrow)$ be a GMLTS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MB}} s_2 \implies \mathrm{E}(s_1) = \mathrm{E}(s_2)$$

■

Lemma 8.5. Let $(S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\ge 0}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2 \implies \mathrm{E}(s_1) = \mathrm{E}(s_2)$$ ∎

Lemma 8.6. Let $(S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\ge 0}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}} s_2 \implies \mathrm{E}(s_1) = \mathrm{E}(s_2)$$ ∎

Theorem 8.7. Let $(S, A, \longrightarrow)$ be a GMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\ge 0}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MB}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$$ ∎

Theorem 8.8. Let $(S, A, \longrightarrow)$ be a GMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\ge 0}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MB}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}} s_2$$ ∎

8.2. Trace Equivalence

Trace equivalence for GMLTS models [128, 16] compares the probability with which two fully stochastic processes perform computations labeled with the same traces within the same amounts of time. In order to formalize this for a GMLTS $(S, A, \longrightarrow)$, as in the GPLTS case (see Sect. 5.2) given $s \in S$ we denote by $\mathcal{C}_{\mathrm{fin}}(s)$ the set of finite-length computations of s and by $|c|$ the length of any $c \in \mathcal{C}_{\mathrm{fin}}(s)$. The probability of executing $c \in \mathcal{C}_{\mathrm{fin}}(s)$ is the product of the *rate-based* execution probabilities of the transitions occurring in c:[11]

$$prob(c) \; = \; \begin{cases} 1 & \text{if } |c| = 0 \\ \frac{\lambda}{\mathrm{E}(s)} \cdot prob(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

which is lifted to $C \subseteq \mathcal{C}_{\mathrm{fin}}(s)$ as follows:

$$prob(C) \; = \; \sum_{c \in C} prob(c)$$

whenever C is finite and all of its computations are independent of each other, i.e., none of them is a proper prefix of one of the others.

In addition to the execution probability, in this stochastic setting we also need to formalize the duration of a finite-length computation. The end-to-end average duration of $c \in \mathcal{C}_{\mathrm{fin}}(s)$ is the sum of the average sojourn times in the states traversed by c:

$$time_{\mathrm{a,ete}}(c) \; = \; \begin{cases} 0 & \text{if } |c| = 0 \\ \frac{1}{\mathrm{E}(s)} + time_{\mathrm{a,ete}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

and we denote by $C_{\le t}$ the set of computations in $C \subseteq \mathcal{C}_{\mathrm{fin}}(s)$ whose end-to-end average duration is not greater than $t \in \mathbb{R}_{\ge 0}$. In contrast, the step-by-step average duration of $c \in \mathcal{C}_{\mathrm{fin}}(s)$ is the sequence of the average sojourn times in the states traversed by c:

$$time_{\mathrm{a,sbs}}(c) \; = \; \begin{cases} \varepsilon & \text{if } |c| = 0 \\ \frac{1}{\mathrm{E}(s)} \circ time_{\mathrm{a,sbs}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

and we denote by $C_{\le \theta}$ the set of computations in $C \subseteq \mathcal{C}_{\mathrm{fin}}(s)$ whose step-by-step average duration is not greater than $\theta \in (\mathbb{R}_{\ge 0})^*$, i.e., $C_{\le \theta} = \{c \in C \mid |c| \le |\theta| \wedge \forall i = 1, \dots, |c|.\, time_{\mathrm{a,sbs}}(c)[i] \le \theta[i]\}$.

As in Sect. 5.2, indicating with $trace(c)$ the sequence of actions labeling the transitions occurring in $c \in \mathcal{C}_{\mathrm{fin}}(s)$, we say that c is compatible with $\alpha \in A^*$ iff $trace(c) = \alpha$. Given $t \in \mathbb{R}_{\ge 0}$ and $\theta \in (\mathbb{R}_{\ge 0})^*$, we denote by $\mathcal{CC}_{\le t}(s, \alpha)$ the set of computations in $\mathcal{C}_{\mathrm{fin}}(s)$ that are compatible with α whose end-to-end average duration is not greater than t and by $\mathcal{CC}_{\le \theta}(s, \alpha)$ the set of computations in $\mathcal{C}_{\mathrm{fin}}(s)$ that are compatible with α whose step-by-step average duration is not greater than θ.

Definition 8.9. Let $(S, A, \longrightarrow)$ be a GMLTS. We say that $s_1, s_2 \in S$ are end-to-end Markovian trace equivalent, written $s_1 \sim_{\mathrm{MTr,ete}} s_2$, iff for all traces $\alpha \in A^*$ and amounts of time $t \in \mathbb{R}_{\ge 0}$:

$$prob(\mathcal{CC}_{\le t}(s_1, \alpha)) \; = \; prob(\mathcal{CC}_{\le t}(s_2, \alpha))$$ ∎

[11] With abuse of notation, we use the same name *prob* employed in the GPLTS case as no confusion arises.

Definition 8.10. Let $(S, A, \longrightarrow)$ be a GMLTS. We say that $s_1, s_2 \in S$ are step-by-step Markovian trace equivalent, written $s_1 \sim_{\mathrm{MTr,sbs}} s_2$, iff for all traces $\alpha \in A^*$ and sequences of amounts of time $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob(\mathcal{CC}_{\leq \theta}(s_1, \alpha)) \ = \ prob(\mathcal{CC}_{\leq \theta}(s_2, \alpha)) \qquad \blacksquare$$

It is worth observing that $\sim_{\mathrm{MTr,ete}}$ and $\sim_{\mathrm{MTr,sbs}}$ do not coincide. In fact, the latter is finer than the former, because it is somehow able to keep track of the time instants at which the various actions of a trace start/complete their execution. As an example, if we consider the following two GMLTS models:

where $\lambda < \mu$ and $b \neq d$, it turns out that $s_1 \sim_{\mathrm{MTr,ete}} s_2$ while $s_1 \not\sim_{\mathrm{MTr,sbs}} s_2$ because $prob(\mathcal{CC}_{\leq \theta}(s_1, \alpha)) = \frac{1}{2} \neq 0 = prob(\mathcal{CC}_{\leq \theta}(s_2, \alpha))$ when $\theta = \frac{1}{2 \cdot \gamma} \circ \frac{1}{\lambda} \circ \frac{1}{\mu}$ and $\alpha = g \circ a \circ b$. We now prove that $\sim_{\mathrm{MTr,ete}}$ is the same as $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete}}}$ and that $\sim_{\mathrm{MTr,sbs}}$ is the same as $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs}}}$, from which it follows that $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete}}}$ and $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs}}}$ do not coincide either.

Theorem 8.11. Let $(S, A, \longrightarrow)$ be a GMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MTr,ete}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete}}} s_2 \qquad \blacksquare$$

Theorem 8.12. Let $(S, A, \longrightarrow)$ be a GMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MTr,sbs}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs}}} s_2 \qquad \blacksquare$$

8.3. Testing Equivalence

Testing equivalence for GMLTS models [22, 16, 17] compares the probability with which two fully stochastic processes pass the same tests within the same amounts of time. Like in the nondeterministic and probabilistic cases, each of the two processes is run in parallel with any test by enforcing synchronization on any action name, with tests being formalized as GMLTS models equipped with a success state.[12] We also extend to interaction systems the notation for the rate-based execution probability and the average duration of computations introduced in Sect. 8.2.

Definition 8.13. A fully stochastic test is a finite-state, acyclic, and finitely-branching GMLTS $\mathcal{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathcal{T}$ is successful iff its last state is ω. $\blacksquare$

Definition 8.14. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be a GMLTS and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a fully stochastic test. The interaction system of $\mathcal{L}$ and $\mathcal{T}$ is the GMLTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = (S \times O, A, \longrightarrow)$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.
- The transition relation $\longrightarrow \ \subseteq (S \times O) \times A \times \mathbb{R}_{>0} \times (S \times O)$ is such that $(s, o) \xrightarrow{a,\lambda} (s', o')$ iff $s \xrightarrow{a,\lambda_1}_{\mathcal{L}} s'$ and $o \xrightarrow{a,\lambda_2}_{\mathcal{T}} o'$ with $\lambda = \lambda_1 \cdot \lambda_2$.[13] We say that a computation of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ is successful iff its last configuration is successful. Given $s \in S$, $o \in O$, $t \in \mathbb{R}_{\geq 0}$, and $\theta \in (\mathbb{R}_{\geq 0})^*$, we denote by $\mathcal{SC}_{\leq t}(s, o)$ the set of successful computations in $\mathcal{I}(\mathcal{L}, \mathcal{T})$ from (s, o) whose end-to-end average duration is not greater than t and by $\mathcal{SC}_{\leq \theta}(s, o)$ the set of successful computations in $\mathcal{I}(\mathcal{L}, \mathcal{T})$ from (s, o) whose step-by-step average duration is not greater than θ. $\blacksquare$

[12] To be precise, the tests considered in [22] for the end-to-end case and in [16] for the step-by-step case were formalized as RPLTS models in a generative-reactive synchronization framework.

[13] The interested reader is referred to [72] for a survey of meaningful operations for computing the rate of the synchronization of two exponentially timed transitions.

Definition 8.15. Let $(S, A, \longrightarrow)$ be a GMLTS. We say that $s_1, s_2 \in S$ are end-to-end Markovian testing equivalent, written $s_1 \sim_{\rm MTe,ete} s_2$, iff for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all amounts of time $t \in \mathbb{R}_{\geq 0}$:

$$prob(\mathcal{SC}_{\leq t}(s_1, o)) \;=\; prob(\mathcal{SC}_{\leq t}(s_2, o)) \qquad \blacksquare$$

Definition 8.16. Let $(S, A, \longrightarrow)$ be a GMLTS. We say that $s_1, s_2 \in S$ are step-by-step Markovian testing equivalent, written $s_1 \sim_{\rm MTe,sbs} s_2$, iff for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all sequences of amounts of time $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob(\mathcal{SC}_{\leq \theta}(s_1, o)) \;=\; prob(\mathcal{SC}_{\leq \theta}(s_2, o)) \qquad \blacksquare$$

Similar to Sect. 8.2, $\sim_{\rm MTe,ete}$ and $\sim_{\rm MTe,sbs}$ do not coincide because the latter is finer than the former. As an example, if we consider again the following two GMLTS models:

where $\lambda < \mu$ and $b \neq d$, it turns out that $s_1 \sim_{\rm MTe,ete} s_2$ while $s_1 \not\sim_{\rm MTe,sbs} s_2$ because $prob(\mathcal{SC}_{\leq \theta}(s_1, o)) = \frac{1}{2} \neq 0 = prob(\mathcal{SC}_{\leq \theta}(s_2, o))$ when $\theta = \frac{1}{2 \cdot \gamma} \circ \frac{1}{\lambda} \circ \frac{1}{\mu}$ and o is the initial state of a fully stochastic test having a single computation that is labeled with $\alpha = g \circ a \circ b$ and reaches ω. We now prove that $\sim_{\rm MTe,ete}$ is the same as $\sim_{{\rm Te},\mathcal{M}_{\rm ete}}$ and $\sim_{\rm MTe,sbs}$ is the same as $\sim_{{\rm Te},\mathcal{M}_{\rm sbs}}$, from which it follows that $\sim_{{\rm Te},\mathcal{M}_{\rm ete}}$ and $\sim_{{\rm Te},\mathcal{M}_{\rm sbs}}$ do not coincide either.

Consistent with Def. 8.14, we denote by $\mathrm{RM} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ the rate multiplication function defined by letting $\mathrm{RM}(\lambda_1, \lambda_2) = \lambda_1 \cdot \lambda_2$, which is 0-preserving and argument-injective. Given an $\mathbb{R}_{\geq 0}$-ULTraS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ and an $\mathbb{R}_{\geq 0}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$, for all configurations (s, o) of their interaction system $\mathcal{I}^{\rm RM}(\mathcal{U}, \mathcal{O})$ and for all $a \in A$ we let $(s, o) \xrightarrow{a} \mathcal{D}$ iff $s \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_1$ and $o \xrightarrow{a}_{\mathcal{O}} \mathcal{D}_2$ with $\mathcal{D}(s', o') = \mathrm{RM}(\mathcal{D}_1(s'), \mathcal{D}_2(o'))$ for each $(s', o') \in S \times O$.

In the following, given $s \in S$, $o \in O$, $\alpha \in A^*$, $t \in \mathbb{R}_{\geq 0}$, and $\theta \in (\mathbb{R}_{\geq 0})^*$ where S is the set of states of a GMLTS and O is the set of states of a fully stochastic test, we denote by $\mathcal{SCC}_{\leq t}(s, o, \alpha)$ the set of computations in $\mathcal{SC}_{\leq t}(s, o)$ that are compatible with α and by $\mathcal{SCC}_{\leq \theta}(s, o, \alpha)$ the set of computations in $\mathcal{SC}_{\leq \theta}(s, o)$ that are compatible with α.

Lemma 8.17. Let $(S, A, \longrightarrow)$ be a GMLTS and $s_1, s_2 \in S$. Then $s_1 \sim_{\rm MTe,ete} s_2$ iff for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$prob(\mathcal{SCC}_{\leq t}(s_1, o, \alpha)) \;=\; prob(\mathcal{SCC}_{\leq t}(s_2, o, \alpha)) \qquad \blacksquare$$

Lemma 8.18. Let $(S, A, \longrightarrow)$ be a GMLTS and $s_1, s_2 \in S$. Then $s_1 \sim_{\rm MTe,sbs} s_2$ iff for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob(\mathcal{SCC}_{\leq \theta}(s_1, o, \alpha)) \;=\; prob(\mathcal{SCC}_{\leq \theta}(s_2, o, \alpha)) \qquad \blacksquare$$

Theorem 8.19. Let $(S, A, \longrightarrow)$ be a GMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTraS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\rm MTe,ete} s_2 \iff s_1 \sim_{{\rm Te},\mathcal{M}^{\rm RM}_{\rm ete}} s_2 \qquad \blacksquare$$

Theorem 8.20. Let $(S, A, \longrightarrow)$ be a GMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTraS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\rm MTe,sbs} s_2 \iff s_1 \sim_{{\rm Te},\mathcal{M}^{\rm RM}_{\rm sbs}} s_2 \qquad \blacksquare$$

9. Equivalences for Reactive Stochastic Processes

In this section, we address a different class of stochastic processes including a limited form of nondeterminism. More precisely, we instantiate the three behavioral equivalences of Sect. 3 for reactive stochastic

$\mathcal{M}_{\mathrm{ete,R}}(s, \alpha, S')(t)$	$=$	$\begin{cases} \int_0^t \mathrm{E}_a(s) \cdot \mathrm{e}^{-\mathrm{E}_a(s) \cdot x} \cdot \sum_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}_a(s)} \cdot \mathcal{M}_{\mathrm{ete,R}}(s', \alpha', S')(t - x)\, \mathrm{d}x & \text{if } \alpha = a \circ \alpha' \text{ and } \mathrm{E}_a(s) > 0 \\ 1 & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ 0 & \text{if } \alpha = a \circ \alpha' \text{ and } \mathrm{E}_a(s) = 0 \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$
$\mathcal{M}_{\mathrm{sbs,R}}(s, \alpha, S')(\theta)$	$=$	$\begin{cases} (1 - \mathrm{e}^{-\mathrm{E}_a(s) \cdot t}) \cdot \sum_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}_a(s)} \cdot \mathcal{M}_{\mathrm{sbs,R}}(s', \alpha', S')(\theta') & \text{if } \alpha = a \circ \alpha' \text{ and } \theta = t \circ \theta' \text{ and } \mathrm{E}_a(s) > 0 \\ 1 & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ 0 & \text{if } \alpha \neq \varepsilon \text{ and } \theta = \varepsilon \\ & \text{or } \alpha = a \circ \alpha' \text{ and } \theta \neq \varepsilon \text{ and } \mathrm{E}_a(s) = 0 \\ & \text{or } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$

Table 5: Measure functions for functional $\mathbb{R}_{\geq 0}$-ULTRAS models representing reactive stochastic processes

processes represented as functional $\mathbb{R}_{[0,1]}$-ULTRAS models (see the first part of Sect. 2.6). Unlike the reactive probabilistic case, in which we have reused the same measure function as the fully probabilistic case, here we need two variants $\mathcal{M}_{\mathrm{ete,R}}$ and $\mathcal{M}_{\mathrm{sbs,R}}$ of the two measure functions $\mathcal{M}_{\mathrm{ete}}$ and $\mathcal{M}_{\mathrm{sbs}}$ that we have respectively introduced in Defs. 8.1 and 8.2 for fully stochastic processes. The reason is that in this reactive setting only conditional exit rates of the form $\mathrm{E}_a(s)$ are meaningful, with $\mathrm{E}_a(s)$ being the sum of the rates of the a-transitions departing from state s. As a consequence, the functions $\mathcal{M}_{\mathrm{ete,R}}(s, \alpha, S')$ and $\mathcal{M}_{\mathrm{sbs,R}}(s, \alpha, S')$ express the probability distribution of performing a computation labeled with trace α that leads to a state in S' from state s within a certain end-to-end or step-by-step deadline, respectively, among all the computations starting at s that are labeled with α.

Definition 9.1. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\geq 0}$-ULTRAS. The end-to-end measure function $\mathcal{M}_{\mathrm{ete,R}} : S \times A^* \times 2^S \to [\mathbb{R}_{\geq 0} \to \mathbb{R}_{[0,1]}]$ for $\mathcal{U}$ is inductively defined in the first part of Table 5, where the associated preorder relation is $\sqsubseteq_{\mathrm{ete}}$ introduced in Def. 8.1. ■

Definition 9.2. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\geq 0}$-ULTRAS. The step-by-step measure function $\mathcal{M}_{\mathrm{sbs,R}} : S \times A^* \times 2^S \to [(\mathbb{R}_{\geq 0})^* \to \mathbb{R}_{[0,1]}]$ for $\mathcal{U}$ is inductively defined in the second part of Table 5, where the associated preorder relation is $\sqsubseteq_{\mathrm{sbs}}$ introduced in Def. 8.2. ■

We now show that the two resulting behavioral equivalences $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,R}}}$ and $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,R}}}$ on functional $\mathbb{R}_{\geq 0}$-ULTRAS models coincide with the bisimulation equivalence defined in the literature for RMLTS models. In contrast, the four remaining behavioral equivalences $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,R}}}$, $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,R}}}$, $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,R}}}$, and $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,R}}}$ on functional $\mathbb{R}_{\geq 0}$-ULTRAS models – where RM is the rate multiplication function introduced in Sect. 8.3 – respectively coincide with the end-to-end and step-by-step trace and testing equivalences for RMLTS models defined for the first time in this paper by analogy with the trace equivalences for GMLTS models and the testing equivalence for RPLTS models respectively discussed in Sects. 8.2 and 6.3.

9.1. Bisimulation Equivalence

Bisimilarity for RMLTS models [95] is defined in the same way as bisimilarity for GMLTS models (see Def. 8.3), with the difference that in an RMLTS $(S, A, \longrightarrow)$ the exit rate $\mathit{rate}_{\mathrm{e}}(s, a, S')$ of a state $s \in S$ with respect to action $a \in A$ and destination $S' \subseteq S$ has a reactive meaning.

Definition 9.3. Let $(S, A, \longrightarrow)$ be an RMLTS. An equivalence relation $\mathcal{B}$ over S is a Markovian bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$:

$$\mathit{rate}_{\mathrm{e}}(s_1, a, C) \; = \; \mathit{rate}_{\mathrm{e}}(s_2, a, C)$$

We say that $s_1, s_2 \in S$ are Markovian bisimilar, written $s_1 \sim_{\mathrm{MB}} s_2$, iff there exists a Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

If we define the *rate-based* reactive exit probability by letting $prob_{\mathrm{e}}(s, a, S') = rate_{\mathrm{e}}(s, a, S')/\mathrm{E}_a(s)$ when $\mathrm{E}_a(s) > 0$, the definition above is equivalent to requiring that, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ it holds that $\mathrm{E}_a(s_1) = \mathrm{E}_a(s_2)$ and for all equivalence classes $C \in S/\mathcal{B}$:

$$prob_{\mathrm{e}}(s_1, a, C) \;=\; prob_{\mathrm{e}}(s_2, a, C)$$

which emphasizes the difference between bisimilarity for RMLTS models and bisimilarity for RPLTS models.

We prove that $\sim_{\mathrm{MB}}$ coincides with $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,R}}}$ and $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,R}}}$, which implies that also for reactive stochastic processes there is no difference between the end-to-end case and the step-by-step case in the bisimulation approach.

Lemma 9.4. Let $(S, A, \longrightarrow)$ be an RMLTS. For all $s_1, s_2 \in S$ and $a \in A$:

$$s_1 \sim_{\mathrm{MB}} s_2 \implies \mathrm{E}_a(s_1) = \mathrm{E}_a(s_2)$$ ■

Lemma 9.5. Let $(S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\geq 0}$-ULTraS. For all $s_1, s_2 \in S$ and $a \in A$:

$$s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,R}}} s_2 \implies \mathrm{E}_a(s_1) = \mathrm{E}_a(s_2)$$ ■

Lemma 9.6. Let $(S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\geq 0}$-ULTraS. For all $s_1, s_2 \in S$ and $a \in A$:

$$s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,R}}} s_2 \implies \mathrm{E}_a(s_1) = \mathrm{E}_a(s_2)$$ ■

Theorem 9.7. Let $(S, A, \longrightarrow)$ be an RMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTraS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MB}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,R}}} s_2$$ ■

Theorem 9.8. Let $(S, A, \longrightarrow)$ be an RMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTraS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MB}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,R}}} s_2$$ ■

9.2. Trace Equivalence

Trace equivalence for RMLTS models can be defined in the same way as trace equivalence for GMLTS models (see Defs. 8.9 and 8.10), with the difference that in a RMLTS $(S, A, \longrightarrow)$ the rate-based execution probability and the average durations of a finite-length computation $c \in \mathcal{C}_{\mathrm{fin}}(s)$ starting from a state $s \in S$ have a reactive meaning and hence are defined as follows:

$$prob_{\mathrm{R}}(c) \;=\; \begin{cases} 1 & \text{if } |c| = 0 \\ \frac{\lambda}{\mathrm{E}_a(s)} \cdot prob_{\mathrm{R}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

$$time_{\mathrm{a,ete,R}}(c) \;=\; \begin{cases} 0 & \text{if } |c| = 0 \\ \frac{1}{\mathrm{E}_a(s)} + time_{\mathrm{a,ete,R}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

$$time_{\mathrm{a,sbs,R}}(c) \;=\; \begin{cases} \varepsilon & \text{if } |c| = 0 \\ \frac{1}{\mathrm{E}_a(s)} \circ time_{\mathrm{a,sbs,R}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

Given a finite set $C \subseteq \mathcal{C}_{\mathrm{fin}}(s)$ of independent computations, in this reactive setting $prob_{\mathrm{R}}(C) = \sum_{c \in C} prob_{\mathrm{R}}(c)$ is well defined only if all the computations in C are compatible with the same trace. Moreover, we denote by $\mathcal{CC}_{\leq t,\mathrm{R}}$ and $\mathcal{CC}_{\leq \theta,\mathrm{R}}$ the variants of $\mathcal{CC}_{\leq t}$ and $\mathcal{CC}_{\leq \theta}$ in which we use $time_{\mathrm{a,ete,R}}$ and $time_{\mathrm{a,sbs,R}}$ in place of $time_{\mathrm{a,ete}}$ and $time_{\mathrm{a,sbs}}$, respectively.

Definition 9.9. Let $(S, A, \longrightarrow)$ be an RMLTS. We say that $s_1, s_2 \in S$ are end-to-end Markovian trace equivalent, written $s_1 \sim_{\mathrm{MTr,ete,R}} s_2$, iff for all traces $\alpha \in A^*$ and amounts of time $t \in \mathbb{R}_{\geq 0}$:

$$prob_{\mathrm{R}}(\mathcal{CC}_{\leq t,\mathrm{R}}(s_1, \alpha)) \;=\; prob_{\mathrm{R}}(\mathcal{CC}_{\leq t,\mathrm{R}}(s_2, \alpha))$$ ■

Definition 9.10. Let $(S, A, \longrightarrow)$ be an RMLTS. We say that $s_1, s_2 \in S$ are step-by-step Markovian trace equivalent, written $s_1 \sim_{\mathrm{MTr,sbs,R}} s_2$, iff for all traces $\alpha \in A^*$ and sequences of amounts of time $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob_{\mathrm{R}}(\mathcal{CC}_{\leq \theta,\mathrm{R}}(s_1, \alpha)) \;=\; prob_{\mathrm{R}}(\mathcal{CC}_{\leq \theta,\mathrm{R}}(s_2, \alpha))$$ ■

Similar to the fully stochastic case, $\sim_{\mathrm{MTr,ete,R}}$ is coarser than $\sim_{\mathrm{MTr,sbs,R}}$ as can be seen by considering two RMLTS models respectively identical to the two GMLTS models of Sect. 8.2. We now prove that $\sim_{\mathrm{MTr,ete,R}}$ is the same as $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,R}}}$ and $\sim_{\mathrm{MTr,sbs,R}}$ is the same as $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,R}}}$, from which it follows that $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,R}}}$ and $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,R}}}$ do not coincide either.

Theorem 9.11. Let $(S, A, \longrightarrow)$ be an RMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MTr,ete,R}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,R}}} s_2$$ ■

Theorem 9.12. Let $(S, A, \longrightarrow)$ be an RMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTRAS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MTr,sbs,R}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,R}}} s_2$$ ■

9.3. Testing Equivalence

Testing equivalence for RMLTS models can be defined in a way similar to testing equivalence for RPLTS models (see Def. 6.7), with the same reactive interpretation of rate-based execution probabilities and average durations of computations as Sect. 9.2 and tests being formalized as RMLTS models equipped with a success state. Then, like in the reactive probabilistic case, we show that the equivalence could have been defined in the same trace-by-trace fashion in which testing equivalence for GMLTS models could have been defined (see Lemmas 8.17 and 8.18).

Definition 9.13. A reactive stochastic test is a finite-state, acyclic, and finitely-branching RMLTS $\mathcal{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathcal{T}$ is successful iff its last state is ω. ■

Definition 9.14. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be an RMLTS and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a reactive stochastic test. The interaction system of $\mathcal{L}$ and $\mathcal{T}$ is the RMLTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = (S \times O, A, \longrightarrow)$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.
- The transition relation $\longrightarrow \subseteq (S \times O) \times A \times \mathbb{R}_{>0} \times (S \times O)$ is such that $(s, o) \xrightarrow{a,\lambda} (s', o')$ iff $s \xrightarrow{a,\lambda_1}_{\mathcal{L}} s'$ and $o \xrightarrow{a,\lambda_2}_{\mathcal{T}} o'$ with $\lambda = \lambda_1 \cdot \lambda_2$.[14] We say that a computation of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ is successful iff its last configuration is successful. Given $s \in S$, $o \in O$, $\alpha \in A^*$, $t \in \mathbb{R}_{\geq 0}$, and $\theta \in (\mathbb{R}_{\geq 0})^*$, we denote by $\mathcal{SCC}_{\leq t,\mathrm{R}}(s, o, \alpha)$ the set of successful computations in $\mathcal{I}(\mathcal{L}, \mathcal{T})$ from (s, o) that are compatible with α whose end-to-end average duration is not grater than t and by $\mathcal{SCC}_{\leq \theta,\mathrm{R}}(s, o, \alpha)$ the set of successful computations in $\mathcal{I}(\mathcal{L}, \mathcal{T})$ from (s, o) that are compatible with α whose step-by-step average duration is not grater than θ. ■

Definition 9.15. Let $(S, A, \longrightarrow)$ be an RMLTS. We say that $s_1, s_2 \in S$ are end-to-end Markovian testing equivalent, written $s_1 \sim_{\mathrm{MTe,ete,R}} s_2$, iff for every reactive stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all amounts of time $t \in \mathbb{R}_{\geq 0}$:

$$\bigsqcup_{\alpha \in \mathit{Tr}_{\max}(s_1, o)} \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq t,\mathrm{R}}(s_1, o, \alpha)) = \bigsqcup_{\alpha \in \mathit{Tr}_{\max}(s_2, o)} \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq t,\mathrm{R}}(s_2, o, \alpha))$$
$$\bigsqcap_{\alpha \in \mathit{Tr}_{\max}(s_1, o)} \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq t,\mathrm{R}}(s_1, o, \alpha)) = \bigsqcap_{\alpha \in \mathit{Tr}_{\max}(s_2, o)} \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq t,\mathrm{R}}(s_2, o, \alpha))$$ ■

Definition 9.16. Let $(S, A, \longrightarrow)$ be an RMLTS. We say that $s_1, s_2 \in S$ are step-by-step Markovian testing equivalent, written $s_1 \sim_{\mathrm{MTe,sbs,R}} s_2$, iff for every reactive stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all sequences of amounts of time $\theta \in (\mathbb{R}_{\geq 0})^*$:

[14] Like in the fully stochastic case, the interested reader is referred to [72] for a survey of meaningful operations for computing the rate of the synchronization of two exponentially timed transitions.

$$\bigsqcup_{\alpha \in \mathit{Tr}_{\max}(s_1,o)} \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq \theta,\mathrm{R}}(s_1,o,\alpha)) \;=\; \bigsqcup_{\alpha \in \mathit{Tr}_{\max}(s_2,o)} \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq \theta,\mathrm{R}}(s_2,o,\alpha))$$
$$\bigsqcap_{\alpha \in \mathit{Tr}_{\max}(s_1,o)} \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq \theta,\mathrm{R}}(s_1,o,\alpha)) \;=\; \bigsqcap_{\alpha \in \mathit{Tr}_{\max}(s_2,o)} \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq \theta,\mathrm{R}}(s_2,o,\alpha)) \quad \blacksquare$$

Similar to the fully stochastic case, $\sim_{\mathrm{MTe,ete,R}}$ is coarser than $\sim_{\mathrm{MTe,sbs,R}}$ as can be seen by considering two RMLTS models respectively identical to the two GMLTS models of Sect. 8.3. We now prove that $\sim_{\mathrm{MTe,ete,R}}$ is the same as $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,R}}}$ and $\sim_{\mathrm{MTe,sbs,R}}$ is the same as $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,R}}}$, from which it follows that $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,R}}}$ and $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,R}}}$ do not coincide either. Like in Sect. 8.3, consistent with Def. 9.14 we employ the rate multiplication function RM in order to build the interaction system of an $\mathbb{R}_{\geq 0}$-ULTRAS and an $\mathbb{R}_{\geq 0}$-observation system.

Lemma 9.17. Let $(S, A, \longrightarrow)$ be an RMLTS and $s_1, s_2 \in S$. Then $s_1 \sim_{\mathrm{MTe,ete,R}} s_2$ iff for every reactive stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:
$$\mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq t,\mathrm{R}}(s_1,o,\alpha)) \;=\; \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq t,\mathrm{R}}(s_2,o,\alpha)) \quad \blacksquare$$

Lemma 9.18. Let $(S, A, \longrightarrow)$ be an RMLTS and $s_1, s_2 \in S$. Then $s_1 \sim_{\mathrm{MTe,sbs,R}} s_2$ iff for every reactive stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:
$$\mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq \theta,\mathrm{R}}(s_1,o,\alpha)) \;=\; \mathit{prob}_{\mathrm{R}}(\mathcal{SCC}_{\leq \theta,\mathrm{R}}(s_2,o,\alpha)) \quad \blacksquare$$

Theorem 9.19. Let $(S, A, \longrightarrow)$ be an RMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTRAS. For all $s_1, s_2 \in S$:
$$s_1 \sim_{\mathrm{MTe,ete,R}} s_2 \iff s_1 \sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,R}}} s_2 \quad \blacksquare$$

Theorem 9.20. Let $(S, A, \longrightarrow)$ be an RMLTS and $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding functional $\mathbb{R}_{\geq 0}$-ULTRAS. For all $s_1, s_2 \in S$:
$$s_1 \sim_{\mathrm{MTe,sbs,R}} s_2 \iff s_1 \sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,R}}} s_2 \quad \blacksquare$$

10. Equivalences for Nondeterministic and Stochastic Processes

In this section, we examine another class of stochastic processes including internal nondeterminism. More precisely, we instantiate the three behavioral equivalences of Sect. 3 for nondeterministic and stochastic processes represented as $\mathbb{R}_{\geq 0}$-ULTRAS models (see the second part of Sect. 2.6). Denoting again by $2^{\mathbb{R}_{[0,1]}}_{\cdot}$ the set of *nonemtpy* subsets of $\mathbb{R}_{[0,1]}$, this is accomplished by introducing two measure functions that associate a suitable $2^{\mathbb{R}_{[0,1]}}_{\cdot}$-valued *function* with every triple composed of a source state s, a trace α, and a set of destination states S' in the end-to-end case and in the step-by-step case, respectively.

The *set* $\mathcal{M}_{\mathrm{ete,N}}(s, \alpha, S')(t)$ computed by the first function contains for each possible way of resolving nondeterminism the probability of performing within time $t \in \mathbb{R}_{\geq 0}$ a computation that is labeled with trace α and leads to a state in S' from state s. The set $\mathcal{M}_{\mathrm{ete,N}}(s, \alpha, S')(t)$ boils down to $\{\mathcal{M}_{\mathrm{ete,R}}(s, \alpha, S')(t)\}$ when there is no internal nondeterminism, i.e., when for every state of the considered $\mathbb{R}_{\geq 0}$-ULTRAS the actions labeling the outgoing transitions are all different from each other. The subscript "ete, N" is a symbolic shorthand for $[\mathbb{R}_{\geq 0} \to 2^{\mathbb{R}_{[0,1]}}_{\cdot}]$. In the following, given a transition $s \xrightarrow{a} \mathcal{D}$ such that $\mathcal{D}(S) = \sum_{s' \in S} \mathcal{D}(s') > 0$, the value $\mathcal{D}(S)$ represents the exit rate of state s determined by that transition.

Definition 10.1. Let $\mathcal{U} = (S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTRAS such that $\mathcal{D}(S) > 0$ for all $(s, a, \mathcal{D}) \in \longrightarrow$. The end-to-end measure function $\mathcal{M}_{\mathrm{ete,N}} : S \times A^* \times 2^S \to [\mathbb{R}_{\geq 0} \to 2^{\mathbb{R}_{[0,1]}}_{\cdot}]$ for $\mathcal{U}$ is inductively defined in the first part of Table 6, where the associated preorder relation is $\sqsubseteq_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}$ introduced in Def. 7.1. $\blacksquare$

The *set* $\mathcal{M}_{\mathrm{sbs,N}}(s, \alpha, S')(\theta)$ computed by the second function contains for each possible way of resolving nondeterminism the probability of performing within time $\theta \in (\mathbb{R}_{\geq 0})^*$ a computation that is labeled with trace α and leads to a state in S' from state s. The set $\mathcal{M}_{\mathrm{sbs,N}}(s, \alpha, S')(\theta)$ boils down to $\{\mathcal{M}_{\mathrm{sbs,R}}(s, \alpha, S')(\theta)\}$ when there is no internal nondeterminism. The subscript "sbs, N" is a symbolic shorthand for $[(\mathbb{R}_{\geq 0})^* \to 2^{\mathbb{R}_{[0,1]}}_{\cdot}]$.

$\mathcal{M}_{\mathrm{ete,N}}(s,\alpha,S')(t) \;=\; \begin{cases} \bigcup\limits_{s \xrightarrow{a} \mathcal{D}} \{\int\limits_0^t \mathcal{D}(S)\cdot \mathrm{e}^{-\mathcal{D}(S)\cdot x} \cdot \sum\limits_{s'\in S} \frac{\mathcal{D}(s')}{\mathcal{D}(S)} \cdot p_{s'}\,\mathrm{d}x \mid p_{s'} \in \mathcal{M}_{\mathrm{ete,N}}(s',\alpha',S')(t-x)\} & \text{if } \alpha = a\circ\alpha' \text{ and } \exists \mathcal{D}\in[S\to\mathbb{R}_{\ge 0}].\, s \xrightarrow{a} \mathcal{D} \\ \{1\} & \text{if } \alpha = \varepsilon \text{ and } s\in S' \\ \{0\} & \text{if } \alpha = a\circ\alpha' \text{ and } \nexists \mathcal{D}\in[S\to\mathbb{R}_{\ge 0}].\, s \xrightarrow{a} \mathcal{D} \\ & \text{or } \alpha=\varepsilon \text{ and } s\notin S' \end{cases}$	
$\mathcal{M}_{\mathrm{sbs,N}}(s,\alpha,S')(\theta) \;=\; \begin{cases} \bigcup\limits_{s \xrightarrow{a} \mathcal{D}} \{(1-\mathrm{e}^{-\mathcal{D}(S)\cdot t}) \cdot \sum\limits_{s'\in S} \frac{\mathcal{D}(s')}{\mathcal{D}(S)} \cdot p_{s'} \mid p_{s'} \in \mathcal{M}_{\mathrm{sbs,N}}(s',\alpha',S')(\theta')\} & \text{if } \alpha = a\circ\alpha' \text{ and } \theta = t\circ\theta' \text{ and } \exists \mathcal{D}\in[S\to\mathbb{R}_{\ge 0}].\, s \xrightarrow{a} \mathcal{D} \\ \{1\} & \text{if } \alpha = \varepsilon \text{ and } s\in S' \\ \{0\} & \text{if } \alpha \neq \varepsilon \text{ and } \theta = \varepsilon \\ & \text{or } \alpha = a\circ\alpha' \text{ and } \theta\neq\varepsilon \text{ and } \nexists \mathcal{D}\in[S\to\mathbb{R}_{\ge 0}].\, s \xrightarrow{a} \mathcal{D} \\ & \text{or } \alpha=\varepsilon \text{ and } s\notin S' \end{cases}$	

Table 6: Measure functions for $\mathbb{R}_{\ge 0}$-ULTRAS models representing nondeterministic and stochastic processes

Definition 10.2. Let $\mathcal{U} = (S, A, \longrightarrow)$ be an $\mathbb{R}_{\ge 0}$-ULTRAS such that $\mathcal{D}(S) > 0$ for all $(s, a, \mathcal{D}) \in \longrightarrow$. The step-by-step measure function $\mathcal{M}_{\mathrm{sbs,N}} : S \times A^* \times 2^S \to [(\mathbb{R}_{\ge 0})^* \to 2^{\mathbb{R}_{[0,1]}}]$ for $\mathcal{U}$ is inductively defined in the second part of Table 6, where the associated preorder relation is $\sqsubseteq_{2^{\mathbb{R}_{[0,1]}}}$ introduced in Def. 7.1. ■

We now show that the six resulting behavioral equivalences $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,N}}}$, $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,N}}}$, $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,N}}}$, $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,N}}}$, $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,N}}}$, and $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,N}}}$ on $\mathbb{R}_{\ge 0}$-ULTRAS models such that $\mathcal{D}(S) > 0$ for all $(s, a, \mathcal{D}) \in \longrightarrow$ – where RM is the rate multiplication function introduced in Sect. 8.3 – respectively coincide with the end-to-end and step-by-step bisimulation, trace, and testing equivalences for NMLTS models defined for the first time in this paper by analogy with the bisimulation, trace, and testing equivalences for NPLTS models respectively discussed in Sects. 7.1, 7.2, and 7.3. Similar to the generative and reactive stochastic cases, we will see that $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,N}}}$ and $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,N}}}$ coincide, whereas the end-to-end trace and testing equivalences are respectively different from the step-by-step trace and testing equivalences.

10.1. Bisimulation Equivalence

Bisimilarity for NMLTS models can be defined in a way similar to bisimilarity for NPLTS models (see Def. 7.3) by combining bisimilarity for fully nondeterministic processes with bisimilarity for fully or reactive stochastic processes. Given an NMLTS $(S, A, \longrightarrow)$, $\mathcal{D} \in [S \to \mathbb{R}_{\ge 0}]$, and $S' \subseteq S$, in the following we let again $\mathcal{D}(S') = \sum_{s' \in S'} \mathcal{D}(s')$. Unlike bisimilarity for NPLTS models, in the case of NMLTS models it is not sufficient to require that $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$ when considering two matching transitions $s_1 \xrightarrow{a} \mathcal{D}_1$ and $s_2 \xrightarrow{a} \mathcal{D}_2$. The reason is that, different from RMLTS models, in this setting with internal nondeterminism a state has, with respect to a given action, as many conditional exit rates as there are outgoing transitions labeled with that action. Therefore, it is not necessarily the case that the two matching transitions above result in the same conditional exit rate for their two source states, i.e., $\mathcal{D}_1(S) = \mathcal{D}_2(S)$. On the other hand, we are guaranteed that s_1 and s_2 reach through those two transitions a state in $\bigcup \mathcal{G}$ with the same probability only if $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$ *and* $\mathcal{D}_1(S) = \mathcal{D}_2(S)$.

Definition 10.3. Let $(S, A, \longrightarrow)$ be an NMLTS. An equivalence relation $\mathcal{B}$ over S is a Markovian bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s_1 \xrightarrow{a} \mathcal{D}_1$ implies $s_2 \xrightarrow{a} \mathcal{D}_2$ with $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$ and $\mathcal{D}_1(S) = \mathcal{D}_2(S)$. We say that $s_1, s_2 \in S$ are Markovian bisimilar, written $s_1 \sim_{\mathrm{MB,N}} s_2$, iff there exists a Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

Theorem 10.4. Let $(S, A, \longrightarrow)$ be an NMLTS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MB,N}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,N}}} s_2 \qquad \blacksquare$$

Theorem 10.5. Let $(S, A, \longrightarrow)$ be an NMLTS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MB,N}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,N}}} s_2 \qquad \blacksquare$$

10.2. Trace Equivalence

Trace equivalence for NMLTS models can be defined in a way similar to trace equivalence for NPLTS models (see Def. 7.7) by combining trace equivalence for fully nondeterministic processes with trace equivalences for fully or reactive stochastic processes. This is accomplished by extending to NMLTS models the notion of resolution via deterministic scheduler introduced for NPLTS models (see Def. 7.5). In the following, given a state s of an NMLTS, the corresponding state z_s of a resolution $\mathcal{Z} \in Res(s)$, $t \in \mathbb{R}_{\geq 0}$, and $\theta \in (\mathbb{R}_{\geq 0})^*$, we denote by $\mathcal{CC}_{\leq t}(z_s, \alpha)$ the set of computations in $\mathcal{C}_{\mathrm{fin}}(z_s)$ that are compatible with $\alpha \in A^*$ whose end-to-end average duration is not greater than t and by $\mathcal{CC}_{\leq \theta}(z_s, \alpha)$ the set of computations in $\mathcal{C}_{\mathrm{fin}}(z_s)$ that are compatible with $\alpha \in A^*$ whose step-by-step average duration is not greater than θ.

Definition 10.6. Let $(S, A, \longrightarrow)$ be an NMLTS. We say that $s_1, s_2 \in S$ are end-to-end Markovian trace equivalent, written $s_1 \sim_{\mathrm{MTr,ete,N}} s_2$, iff for all traces $\alpha \in A^*$ and amounts of time $t \in \mathbb{R}_{\geq 0}$:

- For each resolution $\mathcal{Z}_1 \in Res(s_1)$ there exists a resolution $\mathcal{Z}_2 \in Res(s_2)$ such that:
$$prob(\mathcal{CC}_{\leq t}(z_{s_1}, \alpha)) \; = \; prob(\mathcal{CC}_{\leq t}(z_{s_2}, \alpha))$$
- For each resolution $\mathcal{Z}_2 \in Res(s_2)$ there exists a resolution $\mathcal{Z}_1 \in Res(s_1)$ such that:
$$prob(\mathcal{CC}_{\leq t}(z_{s_2}, \alpha)) \; = \; prob(\mathcal{CC}_{\leq t}(z_{s_1}, \alpha)) \qquad \blacksquare$$

Definition 10.7. Let $(S, A, \longrightarrow)$ be an NMLTS. We say that $s_1, s_2 \in S$ are step-by-step Markovian trace equivalent, written $s_1 \sim_{\mathrm{MTr,sbs,N}} s_2$, iff for all traces $\alpha \in A^*$ and sequences of amounts of time $\theta \in (\mathbb{R}_{\geq 0})^*$:

- For each resolution $\mathcal{Z}_1 \in Res(s_1)$ there exists a resolution $\mathcal{Z}_2 \in Res(s_2)$ such that:
$$prob(\mathcal{CC}_{\leq \theta}(z_{s_1}, \alpha)) \; = \; prob(\mathcal{CC}_{\leq \theta}(z_{s_2}, \alpha))$$
- For each resolution $\mathcal{Z}_2 \in Res(s_2)$ there exists a resolution $\mathcal{Z}_1 \in Res(s_1)$ such that:
$$prob(\mathcal{CC}_{\leq \theta}(z_{s_2}, \alpha)) \; = \; prob(\mathcal{CC}_{\leq \theta}(z_{s_1}, \alpha)) \qquad \blacksquare$$

Similar to the fully and reactive stochastic cases, $\sim_{\mathrm{MTr,ete,N}}$ is coarser than $\sim_{\mathrm{MTr,sbs,N}}$ as can be seen by considering two NMLTS models respectively identical to the two GMLTS models of Sect. 8.2, in which the choice between the two initial transitions becomes nondeterministic. We now prove that $\sim_{\mathrm{MTr,ete,N}}$ is the same as $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,N}}}$ and $\sim_{\mathrm{MTr,sbs,N}}$ is the same as $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,N}}}$, from which it follows that $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,N}}}$ and $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,N}}}$ do not coincide either.

Theorem 10.8. Let $(S, A, \longrightarrow)$ be an NMLTS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MTr,ete,N}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,N}}} s_2 \qquad \blacksquare$$

Theorem 10.9. Let $(S, A, \longrightarrow)$ be an NMLTS. For all $s_1, s_2 \in S$:

$$s_1 \sim_{\mathrm{MTr,sbs,N}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,N}}} s_2 \qquad \blacksquare$$

10.3. Testing Equivalence

Testing equivalence for NMLTS models can be defined in a way similar to testing equivalence for NPLTS models (see Def. 7.12) by adapting to a setting including internal nondeterminism the trace-by-trace characterization of testing equivalences for fully or reactive stochastic processes, with tests being formalized as NMLTS models equipped with a success state.

Definition 10.10. A nondeterministic and stochastic test is a finite-state, acyclic, and finitely-branching NMLTS $\mathcal{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathcal{T}$ is successful iff its last state is ω. ■

Definition 10.11. Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be an NMLTS and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a nondeterministic and stochastic test. The interaction system of $\mathcal{L}$ and $\mathcal{T}$ is the NMLTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = (S \times O, A, \longrightarrow)$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.
- The transition relation $\longrightarrow \subseteq (S \times O) \times A \times [(S \times O) \to \mathbb{R}_{\geq 0}]$ is such that $(s, o) \xrightarrow{a} \mathcal{D}$ iff $s \xrightarrow{a}_{\mathcal{L}} \mathcal{D}_1$ and $o \xrightarrow{a}_{\mathcal{T}} \mathcal{D}_2$ with $\mathcal{D}(s', o') = RM(\mathcal{D}_1(s'), \mathcal{D}_2(o')) = \mathcal{D}_1(s') \cdot \mathcal{D}_2(o')$ for each $(s', o') \in S \times O$.[15] We say that a computation of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ is successful iff its last configuration is successful. Given $s \in S$, $o \in O$, $\mathcal{Z} \in Res(s, o)$, $\alpha \in A^*$, $t \in \mathbb{R}_{\geq 0}$, and $\theta \in (\mathbb{R}_{\geq 0})^*$, we denote by $\mathcal{SCC}_{\leq t}(z_{s,o}, \alpha)$ the set of successful computations in $\mathcal{Z}$ from $z_{s,o}$ that are compatible with α whose end-to-end average duration is not greater than t and by $\mathcal{SCC}_{\leq \theta}(z_{s,o}, \alpha)$ the set of successful computations in $\mathcal{Z}$ from $z_{s,o}$ that are compatible with α whose step-by-step average duration is not greater than θ. ■

Definition 10.12. Let $(S, A, \longrightarrow)$ be an NMLTS. We say that $s_1, s_2 \in S$ are end-to-end Markovian testing equivalent, written $s_1 \sim_{\mathrm{MTe,ete,N}} s_2$, iff for every nondeterministic and stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all traces $\alpha \in A^*$ and amounts of time $t \in \mathbb{R}_{\geq 0}$:

- For each resolution $\mathcal{Z}_1 \in Res(s_1, o)$ there exists a resolution $\mathcal{Z}_2 \in Res(s_2, o)$ such that:
$$prob(\mathcal{SCC}_{\leq t}(z_{s_1,o}, \alpha)) \; = \; prob(\mathcal{SCC}_{\leq t}(z_{s_2,o}, \alpha))$$
- For each resolution $\mathcal{Z}_2 \in Res(s_2, o)$ there exists a resolution $\mathcal{Z}_1 \in Res(s_1, o)$ such that:
$$prob(\mathcal{SCC}_{\leq t}(z_{s_2,o}, \alpha)) \; = \; prob(\mathcal{SCC}_{\leq t}(z_{s_1,o}, \alpha))$$ ■

Definition 10.13. Let $(S, A, \longrightarrow)$ be an NMLTS. We say that $s_1, s_2 \in S$ are step-by-step Markovian testing equivalent, written $s_1 \sim_{\mathrm{MTe,sbs,N}} s_2$, iff for every nondeterministic and stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all traces $\alpha \in A^*$ and sequences of amounts of time $\theta \in (\mathbb{R}_{\geq 0})^*$:

- For each resolution $\mathcal{Z}_1 \in Res(s_1, o)$ there exists a resolution $\mathcal{Z}_2 \in Res(s_2, o)$ such that:
$$prob(\mathcal{SCC}_{\leq \theta}(z_{s_1,o}, \alpha)) \; = \; prob(\mathcal{SCC}_{\leq \theta}(z_{s_2,o}, \alpha))$$
- For each resolution $\mathcal{Z}_2 \in Res(s_2, o)$ there exists a resolution $\mathcal{Z}_1 \in Res(s_1, o)$ such that:
$$prob(\mathcal{SCC}_{\leq \theta}(z_{s_2,o}, \alpha)) \; = \; prob(\mathcal{SCC}_{\leq \theta}(z_{s_1,o}, \alpha))$$ ■

Similar to the fully and reactive stochastic cases, $\sim_{\mathrm{MTe,ete,N}}$ is coarser than $\sim_{\mathrm{MTe,sbs,N}}$ as can be seen by considering two NMLTS models respectively identical to the two GMLTS models of Sect. 8.3, in which the choice between the two initial transitions becomes nondeterministic. We now prove that $\sim_{\mathrm{MTe,ete,N}}$ is the same as $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,N}}}$ and $\sim_{\mathrm{MTe,sbs,N}}$ is the same as $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,N}}}$, from which it follows that $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,N}}}$ and $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,N}}}$ do not coincide either.

Theorem 10.14. Let $(S, A, \longrightarrow)$ be an NMLTS. For all $s_1, s_2 \in S$:
$$s_1 \sim_{\mathrm{MTe,ete,N}} s_2 \iff s_1 \sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,N}}} s_2$$ ■

Theorem 10.15. Let $(S, A, \longrightarrow)$ be an NMLTS. For all $s_1, s_2 \in S$:
$$s_1 \sim_{\mathrm{MTe,sbs,N}} s_2 \iff s_1 \sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,N}}} s_2$$ ■

[15] Like in the fully and reactive stochastic cases, the interested reader is referred to [72] for a survey of meaningful operations for computing the rate of the synchronization of two exponentially timed transitions.

11. Conclusions

In this paper, we have introduced the ULTraS model as an extension of the LTS one. Building on simple probabilistic automata [109] and rate transition systems [48, 49], an ULTraS is defined as a state-transition graph in which the transition relation associates a state reachability distribution, rather than a single target state, with any pair composed of a source state and an action. The one-step reachability values of this distribution are expressed as elements of the support D of a preordered set equipped with a minimum, which is used to represent unreachability. We have shown that, by appropriately selecting D, seven widely used models of nondeterministic, probabilistic, stochastic, and mixed processes can be uniformly described as ULTraS.

We have then reformulated for the ULTraS model the three major notions of behavioral equivalences, namely bisimulation, trace, and testing equivalences. The uniform definition of the three equivalences relies on a measure function based on the support M of another preordered set equipped with a minimum. The measure function expresses the degree of multi-step reachability of a set of states when performing computations labeled with a certain trace. We have proven that the specializations of bisimulation, trace, and testing equivalences obtained by selecting suitable sets M for the measure functions for the various classes of processes represented as ULTraS models are in full agreement with the behavioral equivalences defined in the literature over traditional models for the considered classes of processes, except when combining nondeterminism with probability/stochasticity. On the one hand, this result emphasizes the adequacy of the ULTraS model as a unifying semantic framework; on the other hand, it constitutes a vindication both of the originally proposed models and equivalences and of the new equivalences for mixed models that come out of the general approach.

11.1. Final Considerations

Preordered Sets. Unlike other models such as FuTS [50] where a richer structure in the form of a semi-ring is adopted, in the ULTraS setting we have preferred to rely on simpler structures. We just make use of two distinct preordered sets equipped with a minimum. The first one is used in the basic model to define one-step reachability. The second one is instead used to define measures of multi-step reachability, i.e., observations of computations, when defining equivalences for the different kinds of systems. Having two structures gives us more freedom in defining semantic models and equivalences.

As for the structure of our measure function, in principle we could have used semi-rings. Indeed, like in semi-rings, the measure functions subsumes the existence of an additive and a multiplicative operator (see Sect. 3.1) and many of them compute values in the form of sums of products (see the upper part of Table 1 together with the comment following Def. 4.1, as well as Tables 2, 4, and 5). However, this is not always the case; as an example, consider the measure functions that we have introduced for defining equivalences over nondeterministic and probabilistic/stochastic processes (see Tables 3 and 6) that computes *sets* of sums of products or even convolutions. This fact provides evidence of the difficulty, if not of the impossibility, of developing our unifying framework on the basis of a specific algebraic structure.

Relationships with Known Equivalences. The most interesting outcome of this work is that, when suitably instantiating their measure function, the three behavioral equivalences defined over ULTraS models (bisimulation, trace, and testing) lay the basis for directly capturing most of the behavioral equivalences defined in the literature in the last thirty years. This is shown in Table 7, where the two outer columns indicate corresponding models while the two inner ones indicate corresponding equivalences. Among the equivalences in the second column, only the ones with subscript N or R are new.

In the case of fully nondeterministic processes (first row of Table 7), we observe that – unlike all the other considered classes of processes – it is necessary to introduce two distinct measure functions ($\mathcal{M}_{\mathbb{B},\vee}$ and $\mathcal{M}_{\mathbb{B},\wedge}$ yielding $\mathcal{M}_{\mathbb{B}\times\mathbb{B}}$) for capturing testing equivalence. This is due to the dichotomy between may testing and must testing, which disappears when moving to probabilistic or stochastic processes because the possibility and the necessity of passing tests are subsumed by probability values.

In the case of probabilistic processes (second, third, and fourth rows of Table 7), we note that probabilistic bisimulation, trace, and testing equivalences are defined in the same way in both the generative and

LTS	$\sim_{\rm B}$ $\sim_{\rm Tr}$ $\sim_{\rm Te}$	$\sim_{{\rm B},\mathcal{M}_{\mathbb{B},\vee}}$ $\sim_{{\rm Tr},\mathcal{M}_{\mathbb{B},\vee}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm LC}_{\mathbb{B}\times\mathbb{B}}}$	functional $\mathbb{B}$-ULTraS
GPLTS	$\sim_{\rm PB}$ $\sim_{\rm PTr}$ $\sim_{\rm PTe}$	$\sim_{{\rm B},\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ $\sim_{{\rm Tr},\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm NPM}_{\mathbb{R}_{[0,1]}}}$	functional $\mathbb{R}_{[0,1]}$-ULTraS such that for all $s \in S$ $\sum_{a \in A} \sum_{s' \in S} \mathcal{D}_{s,a}(s') \in \{0,1\}$
RPLTS	$\sim_{\rm PB}$ $\sim_{\rm PTr}$ $\sim_{\rm PTe}$	$\sim_{{\rm B},\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ $\sim_{{\rm Tr},\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm PM}_{\mathbb{R}_{[0,1]}}}$	functional $\mathbb{R}_{[0,1]}$-ULTraS such that for all $s \in S$ and $a \in A$ $\sum_{s' \in S} \mathcal{D}_{s,a}(s') \in \{0,1\}$
NPLTS	$\sim_{\rm PB,N}$ $\sim_{\rm PTr,N}$ $\sim_{\rm PTe,N}$	$\sim_{{\rm B},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}}$ $\sim_{{\rm Tr},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm PM}_{2^{\mathbb{R}_{[0,1]}}}}$	$\mathbb{R}_{[0,1]}$-ULTraS such that for all $(s, a, \mathcal{D}) \in \longrightarrow$ $\sum_{s' \in S} \mathcal{D}(s') = 1$
GMLTS	$\sim_{\rm MB}$ $\sim_{\rm MTr,ete}$ $\sim_{\rm MTr,sbs}$ $\sim_{\rm MTe,ete}$ $\sim_{\rm MTe,sbs}$	$\sim_{{\rm B},\mathcal{M}_{\rm ete}}$ $\sim_{{\rm B},\mathcal{M}_{\rm sbs}}$ $\sim_{{\rm Tr},\mathcal{M}_{\rm ete}}$ $\sim_{{\rm Tr},\mathcal{M}_{\rm sbs}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm RM}_{\rm ete}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm RM}_{\rm sbs}}$	functional $\mathbb{R}_{\geq 0}$-ULTraS
RMLTS	$\sim_{\rm MB}$ $\sim_{\rm MTr,ete,R}$ $\sim_{\rm MTr,sbs,R}$ $\sim_{\rm MTe,ete,R}$ $\sim_{\rm MTe,sbs,R}$	$\sim_{{\rm B},\mathcal{M}_{\rm ete,R}}$ $\sim_{{\rm B},\mathcal{M}_{\rm sbs,R}}$ $\sim_{{\rm Tr},\mathcal{M}_{\rm ete,R}}$ $\sim_{{\rm Tr},\mathcal{M}_{\rm sbs,R}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm RM}_{\rm ete,R}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm RM}_{\rm sbs,R}}$	functional $\mathbb{R}_{\geq 0}$-ULTraS
NMLTS	$\sim_{\rm MB,N}$ $\sim_{\rm MTr,ete,N}$ $\sim_{\rm MTr,sbs,N}$ $\sim_{\rm MTe,ete,N}$ $\sim_{\rm MTe,sbs,N}$	$\sim_{{\rm B},\mathcal{M}_{\rm ete,N}}$ $\sim_{{\rm B},\mathcal{M}_{\rm sbs,N}}$ $\sim_{{\rm Tr},\mathcal{M}_{\rm ete,N}}$ $\sim_{{\rm Tr},\mathcal{M}_{\rm sbs,N}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm RM}_{\rm ete,N}}$ $\sim_{{\rm Te},\mathcal{M}^{\rm RM}_{\rm sbs,N}}$	$\mathbb{R}_{\geq 0}$-ULTraS such that for all $(s, a, \mathcal{D}) \in \longrightarrow$ $\sum_{s' \in S} \mathcal{D}(s') > 0$

Table 7: Summary of results

reactive cases because the different interpretation of probability values is encoded inside the models (see Fig. 1). Therefore, in both cases the same measure function is employed on the ULTraS side; only the synchronization function is slightly different for the two probabilistic testing equivalences. These two testing equivalences were originally defined in a completely different way, but in [25] we have shown that they could have been defined in the same trace-by-trace fashion. It is worth recalling that, for nondeterministic and probabilistic processes, we do not capture the probabilistic bisimulation, trace, and testing equivalences $\sim_{\rm PB,dis}$, $\sim_{\rm PTr,dis}$, and $\sim_{\rm PTe,\sqcup\sqcap}$ originally defined in the literature, but three novel probabilistic equivalences enjoying interesting properties [25, 26]. We observe that, since the NPLTS models on which the original equivalences were introduced are already in the ULTraS format, such equivalences can be straightforwardly recovered by simply applying their definitions to ULTraS models corresponding to NPLTS models. We would also like to remind the reader that the definition of $\sim_{\rm PTe,N}$ is coherent with the trace-by-trace style of probabilistic testing equivalence for generative and reactive probabilistic processes.

In the case of stochastic processes (fifth, sixth, and seventh rows of Table 7), we have addressed both the end-to-end variant and the step-by-step one of the three equivalences. We have seen that these two variants coincide only for bisimulation equivalence. Moreover, we have that, different from the case of probabilistic processes, when moving from generative to reactive stochastic processes only the definition of bisimulation equivalence does not change. This is because the different interpretation of rate values is *not* encoded inside the models (see Fig. 1) and hence needs to be captured by the definition of trace and testing equivalences. As a consequence, two different measure functions for each deadline-related variant of the equivalences are employed on the ULTraS side ($\mathcal{M}_{\rm ete}/\mathcal{M}_{\rm ete,R}$ and $\mathcal{M}_{\rm sbs}/\mathcal{M}_{\rm sbs,R}$), with the same synchronization function being used for the two stochastic testing equivalences. We would like to remind the reader that stochastic trace and testing equivalences for reactive stochastic processes and stochastic bisimulation, trace, and testing equivalences for nondeterministic and stochastic processes had not been studied yet, and that we have defined them by analogy with their probabilistic counterparts.

11.2. Future Work

There are several directions that are worth investigating in the future. First of all, we would like to use the ULTraS model for defining the operational semantics of process description languages of nondeterministic, probabilistic, stochastic, or mixed nature. This should help to establish general properties for the various languages and to assess the relative expressiveness of their operators. This has already been done, by relying on the RTS model, in [48, 49] for the fully stochastic case. Some preliminary work addressing also the fully nondeterministic case and the fully probabilistic case can be found in [24].

What we expect is that the use of the ULTraS model will show that the operational semantics for calculi with quantitative information defined so far is indeed the natural extension of the definition of the operational semantics for calculi with only qualitative information. As a consequence, the former calculi should become more understandable to those people with a process algebraic background who are not familiar with probability and time. The study could lead to:

1. The definition of a generic process calculus for which we can uniformly derive an ULTraS-based operational semantics.
2. The achievement of uniform results for congruence properties as well as equational and logical characterizations of behavioral equivalences.
3. The implementation of a customizable software tool for modeling and analyzing concurrent systems of different nature.

From the generic calculus, it should be possible to retrieve existing calculi by selecting appropriate support sets, suitable behavioral operators, and additional parameters related to the quantitative aspects. For the latter dimension, we will have to consider different alternatives, like including quantities within actions (*integrated quantity approach*) or attaching them as decorations to traditional operators or providing specific operators for them (*orthogonal quantity approach*).

We also plan to extend the uniform definitions of the three behavioral equivalences for ULTraS to models with invisible or silent actions (τ's) whose occurrences have to be abstracted away. In particular, we are interested in uniformly capturing weak variants of bisimulation equivalence, which have already received much attention in the literature. Indeed, weak bisimilarity has been deeply studied not only for nondeterministic processes [68, 91, 93, 126], but also for probabilistic processes [112, 10, 12, 103, 58, 5, 7, 6] and for stochastic processes [108, 69, 30, 89, 20, 18, 19].

Another objective we intend to pursue is studying the extent to which ULTraS can go with encompassing further classes of processes, especially nondeterministic or probabilistic processes with explicit timing. Temporal aspects can be described by means of exponentially distributed random variables, like in the recently proposed Markov automata [59], or arbitrarily distributed random variables. In the former case, the considered model should be easily recoverable in our ULTraS framework by following an approach similar to that taken in [50], where FuTS is used as a semantic model for a significant fragment of the Markov automata process algebra studied in [119]. In the latter case, since the memoryless property can no longer be exploited, it is often necessary to resort to explicit local or global clocks. Processes behavior is then represented through models ranging from enriched labeled transition systems (see [96, 123, 40] and the references therein for deterministic time and nondeterminism, [65, 87, 64] for deterministic time and probability, and [28] for arbitrarily distributed stochastic time) to timed automata [4], probabilistic timed automata [84], and stochastic automata [42].

Finally, we would like to extend the ULTraS framework to the case of transitions of the form $\mathcal{D} \xrightarrow{a} \mathcal{D}'$, where state distributions are allowed not only on the target side, but also on the source side. State distributions can be interpreted as expressing alternatives among (global) states or describing combinations of (local) states. The former interpretation is consistent with the interleaving view of concurrency and, under this interpretation, generalized transitions of the above mentioned form can be viewed as the Kleisli lifting of state-to-state-distribution reachability relations, which have been recently used to define new variants of behavioral equivalences (see, e.g., [67] and the references therein). The latter interpretation instead opens the way to the possibility of representing truly concurrent models such as Petri nets [102]. Using the generalized transition format, a Petri net could be formalized as an $\mathbb{N}$-ULTRAS in which states are Petri net places, transitions are Petri net transitions, and transition sources (resp. targets) are Petri net transition presets (resp. postsets). The value associated with each state in the preset (resp. postset) of a transition is the weight of the corresponding Petri net arc, i.e., the number of tokens to withdraw from that state (resp. to deposit into that state).

Other natural directions for future work would point to coalgebraic characterizations of the ULTRAS model and its equivalences, to new behavioral relations based on approximations and refinements, and to meta-theories for structural operational semantics and its formats. We prefer not to elaborate further on this because the paper is already very long, and also because we are not necessarily the best qualified researchers to investigate these topics; we hope others will pursue these objectives.

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Appendix: Proofs of Results

Proof of Thm. 3.8. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B},\mathcal{M}_M} s_2$. Consider an arbitrary D-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and an arbitrary trace $\alpha \in A^*$. There are two cases:

- If o has no successful computations labeled with α, then:
$$\mathcal{M}_M^{\delta,\mathcal{O}}((s_1, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O})) \;=\; \bot_M \;=\; \mathcal{M}_M^{\delta,\mathcal{O}}((s_2, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$$
- Suppose that o has at least one successful computation labeled with α. Since $s_1 \sim_{\mathrm{B},\mathcal{M}_M} s_2$, there are two subcases:
  - If neither s_1 nor s_2 has a computation labeled with α, then:
  $$\mathcal{M}_M^{\delta,\mathcal{O}}((s_1, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O})) \;=\; \bot_M \;=\; \mathcal{M}_M^{\delta,\mathcal{O}}((s_2, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$$
  - Suppose that both s_1 and s_2 have at least one computation labeled with α. We prove that from $s_1 \sim_{\mathrm{B},\mathcal{M}_M} s_2$ it follows that:
  $$\mathcal{M}_M^{\delta,\mathcal{O}}((s_1, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O})) \;=\; \mathcal{M}_M^{\delta,\mathcal{O}}((s_2, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$$
  by proceeding by induction on $|\alpha|$:
    - $*$ If $|\alpha| = 0$, then by virtue of Def. 3.2 for $i = 1, 2$ the value of $\mathcal{M}_M^{\delta,\mathcal{O}}((s_i, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$ depends only on whether (s_i, o) belongs to $\mathcal{S}^\delta(\mathcal{U}, \mathcal{O})$ or not. Since $\alpha = \varepsilon$ and o has at least one successful computation labeled with α, it turns out that $o = \omega$. Therefore, (s_1, o) and (s_2, o) both belong to $\mathcal{S}^\delta(\mathcal{U}, \mathcal{O})$ and hence:
    $$\mathcal{M}_M^{\delta,\mathcal{O}}((s_1, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O})) \;=\; \mathcal{M}_M^{\delta,\mathcal{O}}((s_2, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$$
    - $*$ Let $|\alpha| = n \in \mathbb{N}_{>0}$ and assume that the result holds for all traces of length $n - 1$. Supposing $\alpha = a \circ \alpha'$, by virtue of Def. 3.2 for $i = 1, 2$ the value of $\mathcal{M}_M^{\delta,\mathcal{O}}((s_i, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$ is defined as a combination of the following set of values:
    $$V_i \;=\; \{\mathcal{M}_M^{\delta,\mathcal{O}}((s'_i, o'), \alpha', \mathcal{S}^\delta(\mathcal{U}, \mathcal{O})) \mid s_i \xrightarrow{a}_{\mathcal{U}} \mathcal{D}_1 \wedge o \xrightarrow{a}_{\mathcal{O}} \mathcal{D}_2\}$$
    each of which is weighted by the reachability in one step of (s'_i, o') through an a-transition. Since $s_1 \sim_{\mathrm{B},\mathcal{M}_M} s_2$, there exists an $\mathcal{M}_M$-bisimulation $\mathcal{B}$ over S that, in particular for the initial a-transitions, guarantees that for all $C \in S/\mathcal{B}$:
    $$\mathcal{M}_M(s_1, a, C) = \mathcal{M}_M(s_2, a, C)$$
    By virtue of Def. 3.2 and the fact that $\mathcal{M}_M^{\delta,\mathcal{O}}$ and δ are functions, we have that for all $C \in S/\mathcal{B}$ and $o' \in O$:

$$\mathcal{M}_M^{\delta,\mathcal{O}}((s_1, o), a, C \times \{o'\}) \;=\; \mathcal{M}_M^{\delta,\mathcal{O}}((s_2, o), a, C \times \{o'\})$$

Moreover, for $i = 1, 2$ it holds that V_i can be rewritten as:

$$V_i = \bigcup_{C \in S/\mathcal{B}} V_i(C)$$

where for all $C \in S/\mathcal{B}$:

$$V_i(C) \;=\; \{\mathcal{M}_M^{\delta,\mathcal{O}}((s'_i, o'), \alpha', \mathcal{S}^\delta(\mathcal{U}, \mathcal{O})) \in V_i \mid s'_i \in C\}$$

By exploiting the induction hypothesis on α', it turns out that for all $C \in S/\mathcal{B}$:

$$V_1(C) \;=\; V_2(C)$$

As a consequence:

$$\mathcal{M}_M^{\delta,\mathcal{O}}((s_1, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O})) \;=\; \mathcal{M}_M^{\delta,\mathcal{O}}((s_2, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}))$$

It thus holds that $s_1 \sim_{\mathrm{Te},\mathcal{M}_M^\delta} s_2$ because of the generality of $\mathcal{O}$ and α.

Suppose now that $s_1 \sim_{\mathrm{Te},\mathcal{M}_M^\delta} s_2$. Given an arbitrary trace $\alpha \in A^*$, consider a D-observation system $\mathcal{O}_\alpha = (O, A, \longrightarrow_{\mathcal{O}_\alpha})$ with initial state $o \in O$ having a single maximal computation, which is labeled with α and terminates in ω. From $s_1 \sim_{\mathrm{Te},\mathcal{M}_M^\delta} s_2$, it follows that:

$$\mathcal{M}_M^{\delta,\mathcal{O}_\alpha}((s_1, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}_\alpha)) \;=\; \mathcal{M}_M^{\delta,\mathcal{O}_\alpha}((s_2, o), \alpha, \mathcal{S}^\delta(\mathcal{U}, \mathcal{O}_\alpha))$$

Due to the structure of $\mathcal{O}_\alpha$ and the fact that δ is $\bot_D$-preserving and argument-injective, it holds that:

$$\mathcal{M}_M(s_1, \alpha, S) \;=\; \mathcal{M}_M(s_2, \alpha, S)$$

and hence $s_1 \sim_{\mathrm{Tr},\mathcal{M}_M} s_2$ because of the generality of α. ∎

Proof of Thm. 4.3. Let $s_1, s_2 \in S$. Assume that $s_1 \sim_{\mathrm{B}} s_2$ due to some bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$:

- whenever $s'_1 \xrightarrow{a} s''_1$, then $s'_2 \xrightarrow{a} s''_2$ with $(s''_1, s''_2) \in \mathcal{B}$;
- whenever $s'_2 \xrightarrow{a} s''_2$, then $s'_1 \xrightarrow{a} s''_1$ with $(s''_1, s''_2) \in \mathcal{B}$.

Without loss of generality, we can suppose that $\mathcal{B}$ is an equivalence relation: should this not be the case, it suffices to take the reflexive and transitive closure of $\mathcal{B}$ as this is still a bisimulation. As a consequence, the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$:

- whenever $s'_1 \xrightarrow{a} s''_1$ with $s''_1 \in C$, then $s'_2 \xrightarrow{a} s''_2$ with $s''_2 \in C$;
- whenever $s'_2 \xrightarrow{a} s''_2$ with $s''_2 \in C$, then $s'_1 \xrightarrow{a} s''_1$ with $s''_1 \in C$;

or equivalently:

- there exists $s''_1 \in C$ such that $s'_1 \xrightarrow{a} s''_1$ iff there exists $s''_2 \in C$ such that $s'_2 \xrightarrow{a} s''_2$.

Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that the existence of $s' \in \bigcup \mathcal{G}$ such that $s \xrightarrow{a} s'$ corresponds to the existence of $s' \in C$ such that $s \xrightarrow{a} s'$ for some $C \in \mathcal{G}$, we immediately derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

- there exists $s''_1 \in \bigcup \mathcal{G}$ such that $s'_1 \xrightarrow{a} s''_1$ iff there exists $s''_2 \in \bigcup \mathcal{G}$ such that $s'_2 \xrightarrow{a} s''_2$.

Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathcal{M}_{\mathbb{B},\vee}(s, a, \bigcup \mathcal{G}) \;=\; \bigvee_{s' \in \bigcup \mathcal{G}} \mathcal{D}_{s,a}(s') \;=\; (\exists s' \in \bigcup \mathcal{G}.\, s \xrightarrow{a} s')$$

we further derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_{\mathbb{B},\vee}(s'_1, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\mathbb{B},\vee}(s'_2, a, \bigcup \mathcal{G})$$

which means that $\mathcal{B}$ is an $\mathcal{M}_{\mathbb{B},\vee}$-bisimulation such that $(s_1, s_2) \in \mathcal{B}$. In other words, $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathbb{B},\vee}} s_2$. ∎

Proof of Thm. 4.5. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{Tr}} s_2$, i.e., assume that for all $\alpha \in A^*$:

- If $s_1 \xRightarrow{\alpha}$, then $s_2 \xRightarrow{\alpha}$.
- If $s_2 \xRightarrow{\alpha}$, then $s_1 \xRightarrow{\alpha}$.

Since for all $s \in S$ and $\alpha \in A^*$ it holds that:

$$(s \xRightarrow{\alpha}) \;=\; \begin{cases} \bigvee_{s' \in S \text{ s.t. } s \xrightarrow{a} s'} (s' \xRightarrow{\alpha'}) & \text{if } \alpha = a \circ \alpha' \text{ and } \exists s' \in S.\, s \xrightarrow{a} s' \\ \top & \text{if } \alpha = \varepsilon \\ \bot & \text{if } \alpha = a \circ \alpha' \text{ and } \nexists s' \in S.\, s \xrightarrow{a} s' \end{cases}$$

and hence:

$$(s \xRightarrow{\alpha}) \;=\; \mathcal{M}_{\mathbb{B},\vee}(s, \alpha, S)$$

we immediately derive that the assumption is equivalent to having that for all $\alpha \in A^*$:

$$\mathcal{M}_{\mathbb{B},\vee}(s_1, \alpha, S) \;=\; \mathcal{M}_{\mathbb{B},\vee}(s_2, \alpha, S)$$

which means that $s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{B},\vee}} s_2$. ■

Proof of Thm. 4.10. The proof is divided into four parts:

- Let $s_1, s_2 \in S$ be such that for every fully nondeterministic test $\mathcal{T} = (O', A, \longrightarrow_{\mathcal{T}})$ with initial state $o' \in O'$:
$$s_1 \text{ may pass } \mathcal{T} \iff s_2 \text{ may pass } \mathcal{T}$$
Then for every $\mathbb{B}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:
$$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_1, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \;=\; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_2, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$
In fact, given $\alpha \in A^*$, there are two cases:
  - If there is no successful computation in $\mathcal{O}$ labeled with α, then:
  $$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_1, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \;=\; \bot \;=\; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_2, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$
  - Suppose that there is at least one successful computation in $\mathcal{O}$ labeled with α. We observe that either both s_1 and s_2 have a computation labeled with α, or neither of them has. In fact, if this were not the case, then only one of them may pass a fully nondeterministic test having a single computation that is labeled with α and reaches ω, thus violating the hypothesis. There are two subcases:
    * If neither s_1 nor s_2 has a computation labeled with α, then:
    $$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_1, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \;=\; \bot \;=\; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_2, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$
    * If both s_1 and s_2 have a computation labeled with α, then:
    $$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_1, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \;=\; \top \;=\; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_2, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$

- Let $s_1, s_2 \in S$ be such that for every $\mathbb{B}$-observation system $\mathcal{O} = (O', A, \longrightarrow_{\mathcal{O}})$ with initial state $o' \in O'$ and for all $\alpha \in A^*$:
$$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_1, o'), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \;=\; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\vee}((s_2, o'), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$
Then for every fully nondeterministic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:
$$s_1 \text{ may pass } \mathcal{T} \iff s_2 \text{ may pass } \mathcal{T}$$
In fact, if this were not the case because of a fully nondeterministic test $\tilde{\mathcal{T}} = (\tilde{O}, A, \longrightarrow_{\tilde{\mathcal{T}}})$ with initial state $\tilde{o} \in \tilde{O}$ such that $(s_1, \tilde{o})$ has at least one successful computation labeled with some trace $\alpha \in A^*$ whereas $(s_2, \tilde{o})$ has no successful computations, then we would have:
$$\mathcal{M}^{\mathrm{LC},\tilde{\mathcal{O}}}_{\mathbb{B},\vee}((s_1, \tilde{o}), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \tilde{\mathcal{O}})) \;=\; \top \;\neq\; \bot \;=\; \mathcal{M}^{\mathrm{LC},\tilde{\mathcal{O}}}_{\mathbb{B},\vee}((s_2, \tilde{o}), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \tilde{\mathcal{O}}))$$
where $\tilde{\mathcal{O}} = (\tilde{O}, A, \longrightarrow_{\tilde{\mathcal{O}}})$ is the $\mathbb{B}$-observation system corresponding to $\tilde{\mathcal{T}}$.

- Let $s_1, s_2 \in S$ be such that for every fully nondeterministic test $\mathcal{T} = (O', A, \longrightarrow_{\mathcal{T}})$ with initial state $o' \in O'$:
$$s_1 \text{ must pass } \mathcal{T} \iff s_2 \text{ must pass } \mathcal{T}$$
Then for every $\mathbb{B}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

$$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_1, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \; = \; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_2, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$

In fact, given $\alpha \in A^*$, there are two cases:

- If there is no computation in $\mathcal{O}$ labeled with α or not all such computations are successful, then:
  $$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_1, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \; = \; \bot \; = \; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_2, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$
- Suppose that there is at least one computation in $\mathcal{O}$ labeled with α and that all such computations are successful. We observe that either both s_1 and s_2 are such that there is a computation labeled with α from each of them and any of their computations labeled with a prefix of α is part of a computation labeled with the entire α, or neither of them is. In fact, if this were not the case, then only one of them should pass a fully nondeterministic test having a single computation that is labeled with α and reaches ω, thus violating the hypothesis. There are two subcases:
  - If neither s_1 nor s_2 is such that there is a computation labeled with α from it and any of its computations labeled with a prefix of α is part of a computation labeled with the entire α, then:
    $$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_1, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \; = \; \bot \; = \; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_2, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$
  - If both s_1 and s_2 are such that there is a computation labeled with α from each of them and any of their computations labeled with a prefix of α is part of a computation labeled with the entire α, then:
    $$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_1, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \; = \; \top \; = \; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_2, o), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$

- Let $s_1, s_2 \in S$ be such that for every $\mathbb{B}$-observation system $\mathcal{O} = (O', A, \longrightarrow_{\mathcal{O}})$ with initial state $o' \in O'$ and for all $\alpha \in A^*$:
  $$\mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_1, o'), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O})) \; = \; \mathcal{M}^{\mathrm{LC},\mathcal{O}}_{\mathbb{B},\wedge}((s_2, o'), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \mathcal{O}))$$
  Then for every fully nondeterministic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:
  $$s_1 \text{ must pass } \mathcal{T} \iff s_2 \text{ must pass } \mathcal{T}$$
  In fact, if this were not the case because of a fully nondeterministic test $\tilde{\mathcal{T}} = (\tilde{O}, A, \longrightarrow_{\tilde{\mathcal{T}}})$ with initial state $\tilde{o} \in \tilde{O}$ such that all the maximal computations from $(s_1, \tilde{o})$ are successful whereas $(s_2, \tilde{o})$ has at least one maximal computation that is not successful, then there would be two possibilities:
  - One possibility is that there exists a trace $\alpha \in A^*$ such that α labels some of the successful computations from $(s_1, \tilde{o})$, but it does not label any successful computation from $(s_2, \tilde{o})$ or one of its prefixes labels an unsuccessful maximal computation from $(s_2, \tilde{o})$. In this case, we would have:
    $$\mathcal{M}^{\mathrm{LC},\tilde{\mathcal{O}}}_{\mathbb{B},\wedge}((s_1, \tilde{o}), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \tilde{\mathcal{O}})) \; = \; \top \neq \bot \; = \; \mathcal{M}^{\mathrm{LC},\tilde{\mathcal{O}}}_{\mathbb{B},\wedge}((s_2, \tilde{o}), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \tilde{\mathcal{O}}))$$
    where $\tilde{\mathcal{O}} = (\tilde{O}, A, \longrightarrow_{\tilde{\mathcal{O}}})$ is the $\mathbb{B}$-observation system corresponding to $\tilde{\mathcal{T}}$.
  - The other possibility is that the set of traces labeling the successful computations from $(s_1, \tilde{o})$ coincides with the set of traces labeling the successful computations from $(s_2, \tilde{o})$ and none of the prefixes of the traces labeling the successful computations from $(s_1, \tilde{o})$ labels an unsuccessful maximal computation from $(s_2, \tilde{o})$. In this case, denoting by α the trace labeling one of the shortest unsuccessful maximal computations from $(s_2, \tilde{o})$, we would have:
    $$\mathcal{M}^{\mathrm{LC},\tilde{\mathcal{O}}}_{\mathbb{B},\wedge}((s_1, \tilde{o}), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \tilde{\mathcal{O}})) \; = \; \bot \neq \top \; = \; \mathcal{M}^{\mathrm{LC},\tilde{\mathcal{O}}}_{\mathbb{B},\wedge}((s_2, \tilde{o}), \alpha, \mathcal{S}^{\mathrm{LC}}(\mathcal{U}, \tilde{\mathcal{O}}))$$
    where $\tilde{\mathcal{O}} = (\tilde{O}, A, \longrightarrow_{\tilde{\mathcal{O}}})$ is the $\mathbb{B}$-observation system corresponding to a variant of $\tilde{\mathcal{T}}$ in which all the computations from $\tilde{o}$ labeled with α are terminated with ω. ■

Proof of Thm. 5.3. Let $s_1, s_2 \in S$. Assume that $s_1 \sim_{\mathrm{PB}} s_2$ due to some probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$:
$$prob_{\mathrm{e}}(s'_1, a, C) \; = \; prob_{\mathrm{e}}(s'_2, a, C)$$
Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:
$$prob_{\mathrm{e}}(s, a, \bigcup \mathcal{G}) \; = \; \sum_{C \in \mathcal{G}} prob_{\mathrm{e}}(s, a, C)$$

we derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$prob_{\mathrm{e}}(s'_1, a, \bigcup \mathcal{G}) \;=\; prob_{\mathrm{e}}(s'_2, a, \bigcup \mathcal{G})$$

Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, a, \bigcup \mathcal{G}) \;=\; \sum_{s' \in \bigcup \mathcal{G}} \mathcal{D}_{s,a}(s') \;=\; prob_{\mathrm{e}}(s, a, \bigcup \mathcal{G})$$

we further derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_1, \alpha, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_2, \alpha, \bigcup \mathcal{G})$$

which means that $\mathcal{B}$ is $\mathcal{M}_{\mathbb{R}_{[0,1]}}$-bisimulation such that $(s_1, s_2) \in \mathcal{B}$. In other words, $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$. ■

Proof of Thm. 5.5. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{PTr}} s_2$, i.e., assume that for all $\alpha \in A^*$:

$$prob(\mathcal{CC}(s_1, \alpha)) \;=\; prob(\mathcal{CC}(s_2, \alpha))$$

Since for all $s \in S$ and $\alpha \in A^*$ it holds that:

$$prob(\mathcal{CC}(s, \alpha)) \;=\; \begin{cases} \sum\limits_{s' \in S} \mathcal{D}_{s,a}(s') \cdot prob(\mathcal{CC}(s', \alpha')) & \text{if } \alpha = a \circ \alpha' \\ 1 & \text{if } \alpha = \varepsilon \end{cases}$$

and hence:

$$prob(\mathcal{CC}(s, \alpha)) \;=\; \mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S)$$

we immediately derive that the assumption is equivalent to having that for all $\alpha \in A^*$:

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s_1, \alpha, S) \;=\; \mathcal{M}_{\mathbb{R}_{[0,1]}}(s_2, \alpha, S)$$

which means that $s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$. ■

Proof of Lemma 5.9. The proof is divided into two parts:

- Suppose that $s_1 \sim_{\mathrm{PTe}} s_2$. Given an arbitrary fully probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and an arbitrary trace $\alpha \in A^*$, consider a variant $\mathcal{T}_\alpha$ of $\mathcal{T}$ in which only the successful computations of $\mathcal{T}$ that are labeled with α reach ω. From $s_1 \sim_{\mathrm{PTe}} s_2$, we derive that:
$$\begin{array}{rcccl} prob(\mathcal{SCC}^{\mathcal{T}}(s_1, o, \alpha)) & = & prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_1, o, \alpha)) & = & \\ & = & prob(\mathcal{SC}^{\mathcal{T}_\alpha}(s_1, o)) & = & \\ & = & prob(\mathcal{SC}^{\mathcal{T}_\alpha}(s_2, o)) & = & \\ & = & prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_2, o, \alpha)) & = & prob(\mathcal{SCC}^{\mathcal{T}}(s_2, o, \alpha)) \end{array}$$
- If for every fully probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:
$$prob(\mathcal{SCC}(s_1, o, \alpha)) \;=\; prob(\mathcal{SCC}(s_2, o, \alpha))$$
then for every fully probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:
$$prob(\mathcal{SC}(s_1, o)) \;=\; \sum_{\alpha \in A^*} prob(\mathcal{SCC}(s_1, o, \alpha)) \;=\; \sum_{\alpha \in A^*} prob(\mathcal{SCC}(s_2, o, \alpha)) \;=\; prob(\mathcal{SC}(s_2, o))$$
which means that $s_1 \sim_{\mathrm{PTe}} s_2$. ■

Proof of Thm. 5.10. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{PTe}} s_2$, i.e., assume that for every fully probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$prob(\mathcal{SC}(s_1, o)) \;=\; prob(\mathcal{SC}(s_2, o))$$

By virtue of Lemma 5.9, the assumption is equivalent to having that for every fully probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

$$prob(\mathcal{SCC}(s_1, o, \alpha)) \;=\; prob(\mathcal{SCC}(s_2, o, \alpha))$$

Let $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ be the $\mathbb{R}_{[0,1]}$-observation system corresponding to an arbitrary fully probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and consider the interaction system $\mathcal{I}^{\mathrm{NPM}}(\mathcal{U}, \mathcal{O})$. Since for all $s \in S$ and $\alpha \in A^*$ it holds that:

$$prob(\mathcal{SCC}(s, o, \alpha)) \;=\; \begin{cases} \sum\limits_{(s', o') \in S \times O} \mathcal{D}_{(s,o),a}(s', o') \cdot prob(\mathcal{SCC}(s', o', \alpha')) & \text{if } \alpha = a \circ \alpha' \\ 1 & \text{if } \alpha = \varepsilon \text{ and } o = \omega \\ 0 & \text{if } \alpha = \varepsilon \text{ and } o \neq \omega \end{cases}$$

and hence:

$$prob(\mathcal{SCC}(s, o, \alpha)) \;=\; \mathcal{M}^{\mathrm{NPM},\mathcal{O}}_{\mathbb{R}_{[0,1]}}((s, o), \alpha, \mathcal{S}^{\mathrm{NPM}}(\mathcal{U}, \mathcal{O}))$$

we immediately derive that the assumption is equivalent to having that for every $\mathbb{R}_{[0,1]}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

$$\mathcal{M}^{\mathrm{NPM},\mathcal{O}}_{\mathbb{R}_{[0,1]}}((s_1, o), \alpha, \mathcal{S}^{\mathrm{NPM}}(\mathcal{U}, \mathcal{O})) \;=\; \mathcal{M}^{\mathrm{NPM},\mathcal{O}}_{\mathbb{R}_{[0,1]}}((s_2, o), \alpha, \mathcal{S}^{\mathrm{NPM}}(\mathcal{U}, \mathcal{O}))$$

which means that $s_1 \sim_{\mathrm{Te},\mathcal{M}^{\mathrm{NPM}}_{\mathbb{R}_{[0,1]}}} s_2$. ■

Proof of Thm. 6.2. The proof is identical to that of Thm. 5.3. ■

Proof of Thm. 6.4. The proof is identical to that of Thm. 5.5. ■

Proof of Lemma 6.8. The proof is divided into two parts:

- Suppose that $s_1 \sim_{\mathrm{PTe}} s_2$. Given an arbitrary reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and an arbitrary trace $\alpha \in A^*$, consider a variant $\mathcal{T}_\alpha$ of $\mathcal{T}$ in which only the successful computations of $\mathcal{T}$ that are labeled with α reach ω. From $s_1 \sim_{\mathrm{PTe}} s_2$, we derive that:
$$\begin{aligned} prob(\mathcal{SCC}^{\mathcal{T}}(s_1, o, \alpha)) &= prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_1, o, \alpha)) = \\ &= \bigsqcup_{\alpha' \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_1, o, \alpha')) = \\ &= \bigsqcup_{\alpha' \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_2, o, \alpha')) = \\ &= prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_2, o, \alpha)) = prob(\mathcal{SCC}^{\mathcal{T}}(s_2, o, \alpha)) \end{aligned}$$
- If for every reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:
$$prob(\mathcal{SCC}(s_1, o, \alpha)) \;=\; prob(\mathcal{SCC}(s_2, o, \alpha))$$
then for every reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:
$$\begin{aligned} \bigsqcup_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) &= \bigsqcup_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha)) \\ \bigsqcap_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) &= \bigsqcap_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha)) \end{aligned}$$
which means that $s_1 \sim_{\mathrm{PTe}} s_2$. ■

Proof of Thm. 6.9. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{PTe}} s_2$, i.e., assume that for every reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$\begin{aligned} \bigsqcup_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) &= \bigsqcup_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha)) \\ \bigsqcap_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) &= \bigsqcap_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha)) \end{aligned}$$

By virtue of Lemma 6.8, the assumption is equivalent to having that for every reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

$$prob(\mathcal{SCC}(s_1, o, \alpha)) \;=\; prob(\mathcal{SCC}(s_2, o, \alpha))$$

Let $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ be the $\mathbb{R}_{[0,1]}$-observation system corresponding to an arbitrary reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and consider the interaction system $\mathcal{I}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})$. Since for all $s \in S$ and $\alpha \in A^*$ it holds that:

$$prob(\mathcal{SCC}(s, o, \alpha)) \;=\; \begin{cases} \sum\limits_{(s', o') \in S \times O} \mathcal{D}_{(s,o),a}(s', o') \cdot prob(\mathcal{SCC}(s', o', \alpha')) & \text{if } \alpha = a \circ \alpha' \\ 1 & \text{if } \alpha = \varepsilon \text{ and } o = \omega \\ 0 & \text{if } \alpha = \varepsilon \text{ and } o \neq \omega \end{cases}$$

and hence:

$$prob(\mathcal{SCC}(s, o, \alpha)) \;=\; \mathcal{M}^{\mathrm{PM},\mathcal{O}}_{\mathbb{R}_{[0,1]}}((s, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O}))$$

we immediately derive that the assumption is equivalent to having that for every $\mathbb{R}_{[0,1]}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

$$\mathcal{M}^{\mathrm{PM},\mathcal{O}}_{\mathbb{R}_{[0,1]}}((s_1, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})) \;=\; \mathcal{M}^{\mathrm{PM},\mathcal{O}}_{\mathbb{R}_{[0,1]}}((s_2, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O}))$$

which means that $s_1 \sim_{\mathrm{Te},\mathcal{M}^{\mathrm{PM}}_{\mathbb{R}_{[0,1]}}} s_2$. ■

Proof of Thm. 7.4. Let $s_1, s_2 \in S$. Assume that $s_1 \sim_{\mathrm{PB,N}} s_2$ due to some probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a} \mathcal{D}_1$ implies $s'_2 \xrightarrow{a} \mathcal{D}_2$ with $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\bigcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{\mathcal{D}_1(\textstyle\bigcup \mathcal{G})\} \subseteq \bigcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{\mathcal{D}_2(\textstyle\bigcup \mathcal{G})\}$$

$$\bigcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{\mathcal{D}_2(\textstyle\bigcup \mathcal{G})\} \subseteq \bigcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{\mathcal{D}_1(\textstyle\bigcup \mathcal{G})\}$$

or equivalently:

$$\bigcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{\mathcal{D}_1(\textstyle\bigcup \mathcal{G})\} = \bigcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{\mathcal{D}_2(\textstyle\bigcup \mathcal{G})\}$$

Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s, a, \bigcup \mathcal{G}) = \{0\}$ when s has no a-transition, otherwise:

$$\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s, a, \textstyle\bigcup \mathcal{G}) = \displaystyle\bigcup_{s \xrightarrow{a} \mathcal{D}} \{\sum_{s' \in \bigcup \mathcal{G}} \mathcal{D}(s')\} = \bigcup_{s \xrightarrow{a} \mathcal{D}} \{\mathcal{D}(\textstyle\bigcup \mathcal{G})\}$$

we derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s'_1, a, \textstyle\bigcup \mathcal{G}) = \mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s'_2, a, \textstyle\bigcup \mathcal{G})$$

which means that $\mathcal{B}$ is an $\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}$-bisimulation such that $(s_1, s_2) \in \mathcal{B}$. In other words, $s_1 \sim_{\mathrm{B},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}} s_2$. ∎

Proof of Thm. 7.8. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{PTr,N}} s_2$, i.e., assume that for all $\alpha \in A^*$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that:
$$\mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) = \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha))$$
- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ such that:
$$\mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha)) = \mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha))$$

Since for all $s \in S$, $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in \mathit{Res}(s)$, and $\alpha \in A^*$ it holds that:

$$\mathit{prob}(\mathcal{CC}(z_s, \alpha)) = \begin{cases} \sum_{z_{s'} \in Z} \mathcal{D}(z_{s'}) \cdot \mathit{prob}(\mathcal{CC}(z_{s'}, \alpha')) & \text{if } \alpha = a \circ \alpha' \text{ and } \exists \mathcal{D} \in [Z \to \mathbb{R}_{[0,1]}].\, z_s \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} \\ 1 & \text{if } \alpha = \varepsilon \\ 0 & \text{if } \alpha = a \circ \alpha' \text{ and } \nexists \mathcal{D} \in [Z \to \mathbb{R}_{[0,1]}].\, z_s \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} \end{cases}$$

and hence:

$$\mathit{prob}(\mathcal{CC}(z_s, \alpha)) \in \mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s, \alpha, S)$$

we immediately derive that the assumption is equivalent to having that for all $\alpha \in A^*$:

$$\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s_1, \alpha, S) \subseteq \mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s_2, \alpha, S)$$
$$\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s_2, \alpha, S) \subseteq \mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s_1, \alpha, S)$$

or equivalently:

$$\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s_1, \alpha, S) = \mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}(s_2, \alpha, S)$$

which means that $s_1 \sim_{\mathrm{Tr},\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}} s_2$. ∎

Proof of Thm. 7.13. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{PTe,N}} s_2$, i.e., assume that for every nondeterministic and probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1, o)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2, o)$ such that:
$$\mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha)) = \mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2, o)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1, o)$ such that:
$$\mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha)) = \mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha))$$

Let $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ be the $\mathbb{R}_{[0,1]}$-observation system corresponding to an arbitrary nondeterministic and probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and consider the interaction system $\mathcal{I}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})$

where $\mathcal{U}$ is the NPLTS under examination. Since for all $s \in S$, $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s, o)$, and $\alpha \in A^*$ it holds that:

$$prob(\mathcal{SCC}(z_{s,o}, \alpha)) \; = \; \begin{cases} \sum\limits_{z_{s',o'} \in Z} \mathcal{D}(z_{s',o'}) \cdot prob(\mathcal{SCC}(z_{s',o'}, \alpha')) & \\ & \text{if } \alpha = a \circ \alpha' \text{ and } \exists \mathcal{D} \in [Z \rightarrow \mathbb{R}_{[0,1]}].\, z_{s,o} \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} \\ 1 & \text{if } \alpha = \varepsilon \text{ and } o = \omega \\ 0 & \text{if } \alpha = a \circ \alpha' \text{ and } \nexists \mathcal{D} \in [Z \rightarrow \mathbb{R}_{[0,1]}].\, z_{s,o} \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} \\ & \text{or } \alpha = \varepsilon \text{ and } o \neq \omega \end{cases}$$

and hence:

$$prob(\mathcal{SCC}(z_{s,o}, \alpha)) \; \in \; \mathcal{M}^{\mathrm{PM},\mathcal{O}}_{2^{\mathbb{R}_{[0,1]}}}((s, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O}))$$

we immediately derive that the assumption is equivalent to having that for every $\mathbb{R}_{[0,1]}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

$$\begin{array}{rcl} \mathcal{M}^{\mathrm{PM},\mathcal{O}}_{2^{\mathbb{R}_{[0,1]}}}((s_1, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})) & \subseteq & \mathcal{M}^{\mathrm{PM},\mathcal{O}}_{2^{\mathbb{R}_{[0,1]}}}((s_2, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})) \\ \mathcal{M}^{\mathrm{PM},\mathcal{O}}_{2^{\mathbb{R}_{[0,1]}}}((s_2, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})) & \subseteq & \mathcal{M}^{\mathrm{PM},\mathcal{O}}_{2^{\mathbb{R}_{[0,1]}}}((s_1, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})) \end{array}$$

or equivalently:

$$\mathcal{M}^{\mathrm{PM},\mathcal{O}}_{2^{\mathbb{R}_{[0,1]}}}((s_1, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O})) \; = \; \mathcal{M}^{\mathrm{PM},\mathcal{O}}_{2^{\mathbb{R}_{[0,1]}}}((s_2, o), \alpha, \mathcal{S}^{\mathrm{PM}}(\mathcal{U}, \mathcal{O}))$$

which means that $s_1 \sim_{\mathrm{Te},\mathcal{M}^{\mathrm{PM}}_{2^{\mathbb{R}_{[0,1]}}}} s_2$. ■

Proof of Lemma 8.4. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{MB}} s_2$. Then it immediately follows that:

$$\mathrm{E}(s_1) \; = \; \sum_{a \in A} \sum_{C \in S/\sim_{\mathrm{MB}}} rate_{\mathrm{e}}(s_1, a, C) \; = \; \sum_{a \in A} \sum_{C \in S/\sim_{\mathrm{MB}}} rate_{\mathrm{e}}(s_2, a, C) \; = \; \mathrm{E}(s_2)$$

■

Proof of Lemma 8.5. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$. Then either $\mathrm{E}(s_1) = 0 = \mathrm{E}(s_2)$, in which case the result trivially holds, or $\mathrm{E}(s_1) > 0$ and $\mathrm{E}(s_2) > 0$, which is the case that we examine. From $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$, it follows that:

$$\sum_{a \in A} \mathcal{M}_{\mathrm{ete}}(s_1, a, S) \; = \; \sum_{a \in A} \sum_{C \in S/\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}} \mathcal{M}_{\mathrm{ete}}(s_1, a, C) \; = \; \sum_{a \in A} \sum_{C \in S/\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}} \mathcal{M}_{\mathrm{ete}}(s_2, a, C) \; = \; \sum_{a \in A} \mathcal{M}_{\mathrm{ete}}(s_2, a, S)$$

Since for all $s \in S$ such that $\mathrm{E}(s) > 0$ and for all $t \in \mathbb{R}_{\geq 0}$ it holds that:

$$\begin{array}{rcl} \sum\limits_{a \in A} \mathcal{M}_{\mathrm{ete}}(s, a, S)(t) & = & \sum\limits_{a \in A} \int\limits_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum\limits_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \, \mathrm{d}x \\ & = & \sum\limits_{a \in A} \sum\limits_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \int\limits_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \, \mathrm{d}x \\ & = & \frac{1}{\mathrm{E}(s)} \cdot \sum\limits_{a \in A} \sum\limits_{s' \in S} \mathcal{D}_{s,a}(s') \cdot (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \\ & = & 1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t} \end{array}$$

we further derive that for all $t \in \mathbb{R}_{\geq 0}$:

$$1 - \mathrm{e}^{-\mathrm{E}(s_1) \cdot t} \; = \; 1 - \mathrm{e}^{-\mathrm{E}(s_2) \cdot t}$$

and hence:

$$\mathrm{E}(s_1) \; = \; \mathrm{E}(s_2)$$

■

Proof of Lemma 8.6. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}} s_2$. Then either $\mathrm{E}(s_1) = 0 = \mathrm{E}(s_2)$, in which case the result trivially holds, or $\mathrm{E}(s_1) > 0$ and $\mathrm{E}(s_2) > 0$, which is the case that we examine. From $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}} s_2$, it follows that:

$$\sum_{a \in A} \mathcal{M}_{\mathrm{sbs}}(s_1, a, S) \; = \; \sum_{a \in A} \sum_{C \in S/\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}}} \mathcal{M}_{\mathrm{sbs}}(s_1, a, C) \; = \; \sum_{a \in A} \sum_{C \in S/\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}}} \mathcal{M}_{\mathrm{sbs}}(s_2, a, C) \; = \; \sum_{a \in A} \mathcal{M}_{\mathrm{sbs}}(s_2, a, S)$$

Since for all $s \in S$ such that $\mathrm{E}(s) > 0$ and for all $\theta = t \circ \theta' \in (\mathbb{R}_{\geq 0})^*$ it holds that:

$$\begin{array}{rcl} \sum\limits_{a \in A} \mathcal{M}_{\mathrm{sbs}}(s, a, S)(\theta) & = & \sum\limits_{a \in A} (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot \sum\limits_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \\ & = & (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot \frac{1}{\mathrm{E}(s)} \cdot \sum\limits_{a \in A} \sum\limits_{s' \in S} \mathcal{D}_{s,a}(s') \\ & = & 1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t} \end{array}$$

we further derive that for all $t \in \mathbb{R}_{\geq 0}$:

$$1 - \mathrm{e}^{-\mathrm{E}(s_1) \cdot t} \; = \; 1 - \mathrm{e}^{-\mathrm{E}(s_2) \cdot t}$$

and hence:

$$\mathrm{E}(s_1) \;=\; \mathrm{E}(s_2) \qquad \blacksquare$$

Proof of Thm. 8.7. Let $s_1, s_2 \in S$. Assume that $s_1 \sim_{\mathrm{MB}} s_2$ due to some Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$:

$$\mathit{rate}_{\mathrm{e}}(s'_1, a, C) \;=\; \mathit{rate}_{\mathrm{e}}(s'_2, a, C)$$

Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathit{rate}_{\mathrm{e}}(s, a, \bigcup \mathcal{G}) \;=\; \sum_{C \in \mathcal{G}} \mathit{rate}_{\mathrm{e}}(s, a, C)$$

we derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathit{rate}_{\mathrm{e}}(s'_1, a, \bigcup \mathcal{G}) \;=\; \mathit{rate}_{\mathrm{e}}(s'_2, a, \bigcup \mathcal{G})$$

Since for all $s \in S$, $a \in A$, $\mathcal{G} \in 2^{S/\mathcal{B}}$, and $t \in \mathbb{R}_{\geq 0}$ it holds that $\mathcal{M}_{\mathrm{ete}}(s, a, \bigcup \mathcal{G})(t) = 0$ when $\mathrm{E}(s) = 0$, otherwise:

$$\begin{array}{rcl}
\mathcal{M}_{\mathrm{ete}}(s, a, \bigcup \mathcal{G})(t) & = & \int\limits_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum\limits_{s' \in \bigcup \mathcal{G}} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \, \mathrm{d}x \\
& = & (\int\limits_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \, \mathrm{d}x) \cdot \frac{1}{\mathrm{E}(s)} \cdot \sum\limits_{s' \in \bigcup \mathcal{G}} \mathcal{D}_{s,a}(s') \\
& = & \frac{1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}}{\mathrm{E}(s)} \cdot \sum\limits_{s' \in \bigcup \mathcal{G}} \mathcal{D}_{s,a}(s') \\
& = & \frac{1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}}{\mathrm{E}(s)} \cdot \mathit{rate}_{\mathrm{e}}(s, a, \bigcup \mathcal{G})
\end{array}$$

we further derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_{\mathrm{ete}}(s'_1, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\mathrm{ete}}(s'_2, a, \bigcup \mathcal{G})$$

which means that $\mathcal{B}$ is an $\mathcal{M}_{\mathrm{ete}}$-bisimulation such that $(s_1, s_2) \in \mathcal{B}$. In other words, $s_1 \sim_{\mathrm{B}, \mathcal{M}_{\mathrm{ete}}} s_2$. Note that $\mathrm{E}(s'_1) = \mathrm{E}(s'_2)$ is guaranteed by Lemma 8.4 (direction $\Longrightarrow$) and Lemma 8.5 (direction $\Longleftarrow$). $\blacksquare$

Proof of Thm. 8.8. Let $s_1, s_2 \in S$. Assume that $s_1 \sim_{\mathrm{MB}} s_2$ due to some Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$:

$$\mathit{rate}_{\mathrm{e}}(s'_1, a, C) \;=\; \mathit{rate}_{\mathrm{e}}(s'_2, a, C)$$

Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathit{rate}_{\mathrm{e}}(s, a, \bigcup \mathcal{G}) \;=\; \sum_{C \in \mathcal{G}} \mathit{rate}_{\mathrm{e}}(s, a, C)$$

we derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathit{rate}_{\mathrm{e}}(s'_1, a, \bigcup \mathcal{G}) \;=\; \mathit{rate}_{\mathrm{e}}(s'_2, a, \bigcup \mathcal{G})$$

Since for all $s \in S$, $a \in A$, $\mathcal{G} \in 2^{S/\mathcal{B}}$, and $\theta \in (\mathbb{R}_{\geq 0})^*$ it holds that $\mathcal{M}_{\mathrm{sbs}}(s, a, \bigcup \mathcal{G})(\theta) = 0$ when $\theta = \varepsilon$ or $\mathrm{E}(s) = 0$, otherwise for $\theta = t \circ \theta'$:

$$\begin{array}{rcl}
\mathcal{M}_{\mathrm{sbs}}(s, a, \bigcup \mathcal{G})(\theta) & = & (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot \sum\limits_{s' \in \bigcup \mathcal{G}} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \\
& = & \frac{1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}}{\mathrm{E}(s)} \cdot \sum\limits_{s' \in \bigcup \mathcal{G}} \mathcal{D}_{s,a}(s') \\
& = & \frac{1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}}{\mathrm{E}(s)} \cdot \mathit{rate}_{\mathrm{e}}(s, a, \bigcup \mathcal{G})
\end{array}$$

we further derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_{\mathrm{sbs}}(s'_1, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\mathrm{sbs}}(s'_2, a, \bigcup \mathcal{G})$$

which means that $\mathcal{B}$ is an $\mathcal{M}_{\mathrm{sbs}}$-bisimulation such that $(s_1, s_2) \in \mathcal{B}$. In other words, $s_1 \sim_{\mathrm{B}, \mathcal{M}_{\mathrm{sbs}}} s_2$. Note that $\mathrm{E}(s'_1) = \mathrm{E}(s'_2)$ is guaranteed by Lemma 8.4 (direction $\Longrightarrow$) and Lemma 8.6 (direction $\Longleftarrow$). $\blacksquare$

Proof of Thm. 8.11. Given $s \in S$, we define the end-to-end duration of $c \in \mathcal{C}_{\mathrm{fin}}(s)$ as the sum of the random variables quantifying the sojourn times in the states traversed by c:

$$\mathit{time}_{\mathrm{d,ete}}(c) \;=\; \begin{cases} \mathit{Det}_0 & \text{if } |c| = 0 \\ \mathit{Exp}_s + \mathit{time}_{\mathrm{d,ete}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

where Det_0 is the random variable equal to 0 with probability 1, while Exp_s is the exponentially distributed random variable with rate $\mathrm{E}(s)$. Moreover, we define the probability distribution of executing a computation in $C \subseteq \mathcal{C}_{\mathrm{fin}}(s)$ within $t \in \mathbb{R}_{\geq 0}$ time units by letting:

$$prob_{\mathrm{d,ete}}(C, t) \;=\; \sum_{c \in C} prob(c) \cdot \Pr\{time_{\mathrm{d,ete}}(c) \leq t\}$$

whenever C is finite and all of its computations are independent of each other.
Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{MTr,ete}} s_2$, i.e., assume that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$prob(\mathcal{CC}_{\leq t}(s_1, \alpha)) \;=\; prob(\mathcal{CC}_{\leq t}(s_2, \alpha))$$

Due to [22, 16], this is equivalent to having that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$prob_{\mathrm{d,ete}}(\mathcal{CC}(s_1, \alpha), t) \;=\; prob_{\mathrm{d,ete}}(\mathcal{CC}(s_2, \alpha), t)$$

Since for all $s \in S$, $\alpha \in A^*$, and $t \in \mathbb{R}_{\geq 0}$ it holds that:

$$prob_{\mathrm{d,ete}}(\mathcal{CC}(s, \alpha), t) \;=\; \begin{cases} \sum\limits_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \int\limits_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot prob_{\mathrm{d,ete}}(\mathcal{CC}(s', \alpha'), t - x)\, \mathrm{d}x & \\ & \text{if } \alpha = a \circ \alpha' \text{ and } \mathrm{E}(s) > 0 \\ 1 & \text{if } \alpha = \varepsilon \\ 0 & \text{if } \alpha \neq \varepsilon \text{ and } \mathrm{E}(s) = 0 \end{cases}$$

and hence:

$$prob_{\mathrm{d,ete}}(\mathcal{CC}(s, \alpha), t) \;=\; \mathcal{M}_{\mathrm{ete}}(s, \alpha, S)(t)$$

we immediately derive that the assumption is equivalent to having that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$\mathcal{M}_{\mathrm{ete}}(s_1, \alpha, S)(t) \;=\; \mathcal{M}_{\mathrm{ete}}(s_2, \alpha, S)(t)$$

which means that $s_1 \sim_{\mathrm{Tr}, \mathcal{M}_{\mathrm{ete}}} s_2$. ■

Proof of Thm. 8.12. Given $s \in S$, we define the step-by-step duration of $c \in \mathcal{C}_{\mathrm{fin}}(s)$ as the sequence of the random variables quantifying the average sojourn times in the states traversed by c:

$$time_{\mathrm{d,sbs}}(c) \;=\; \begin{cases} Det_0 & \text{if } |c| = 0 \\ Exp_s \circ time_{\mathrm{d,sbs}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

where, as in the proof of Thm. 8.11, Det_0 is the random variable equal to 0 with probability 1, while Exp_s is the exponentially distributed random variable with rate $\mathrm{E}(s)$. Moreover, we define the probability distribution of executing a computation in $C \subseteq \mathcal{C}_{\mathrm{fin}}(s)$ within a sequence $\theta \in (\mathbb{R}_{\geq 0})^*$ of time units by letting:

$$\begin{aligned} prob_{\mathrm{d,sbs}}(C, \theta) &= \sum_{c \in C}^{|c| \leq |\theta|} prob(c) \cdot \prod_{i=1}^{|c|} \Pr\{time_{\mathrm{d,sbs}}(c)[i] \leq \theta[i]\} \\ &= \sum_{c \in C}^{|c| \leq |\theta|} prob(c) \cdot \prod_{i=1}^{|c|} (1 - \mathrm{e}^{-\theta[i] / time_{\mathrm{a,sbs}}(c)[i]}) \end{aligned}$$

whenever C is finite and all of its computations are independent of each other.
Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{MTr,sbs}} s_2$, i.e., assume that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob(\mathcal{CC}_{\leq \theta}(s_1, \alpha)) \;=\; prob(\mathcal{CC}_{\leq \theta}(s_2, \alpha))$$

Due to [16], this is equivalent to having that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob_{\mathrm{d,sbs}}(\mathcal{CC}(s_1, \alpha), \theta) \;=\; prob_{\mathrm{d,sbs}}(\mathcal{CC}(s_2, \alpha), \theta)$$

Since for all $s \in S$, $\alpha \in A^*$, and $\theta \in (\mathbb{R}_{\geq 0})^*$ it holds that:

$$prob_{\mathrm{d,sbs}}(\mathcal{CC}(s, \alpha), \theta) \;=\; \begin{cases} \sum\limits_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot prob_{\mathrm{d,sbs}}(\mathcal{CC}(s', \alpha'), \theta') & \\ & \text{if } \alpha = a \circ \alpha' \text{ and } \theta = t \circ \theta' \text{ and } \mathrm{E}(s) > 0 \\ 1 & \text{if } \alpha = \varepsilon \\ 0 & \text{if } \alpha \neq \varepsilon \text{ and } \theta = \varepsilon \\ & \text{or } \alpha \neq \varepsilon \text{ and } \theta \neq \varepsilon \text{ and } \mathrm{E}(s) = 0 \end{cases}$$

and hence:

$$prob_{\mathrm{d,sbs}}(\mathcal{CC}(s, \alpha), \theta) \;=\; \mathcal{M}_{\mathrm{sbs}}(s, \alpha, S)(\theta)$$

we immediately derive that the assumption is equivalent to having that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$\mathcal{M}_{\mathrm{sbs}}(s_1, \alpha, S)(\theta) \;=\; \mathcal{M}_{\mathrm{sbs}}(s_2, \alpha, S)(\theta)$$

which means that $s_1 \sim_{\mathrm{Tr}, \mathcal{M}_{\mathrm{sbs}}} s_2$. ■

Proof of Lemma 8.17. The proof is identical to that of Lemma 5.9 up to the use of $\sim_{\mathrm{MTe,ete}}$, fully stochastic tests, $\mathcal{SCC}_{\leq t}$, and $\mathcal{SC}_{\leq t}$ in place of $\sim_{\mathrm{PTe}}$, fully probabilistic tests, $\mathcal{SCC}$, and $\mathcal{SC}$. ■

Proof of Lemma 8.18. The proof is identical to that of Lemma 5.9 up to the use of $\sim_{\text{MTe,sbs}}$, fully stochastic tests, $\mathcal{SCC}_{\leq\theta}$, and $\mathcal{SC}_{\leq\theta}$ in place of $\sim_{\text{PTe}}$, fully probabilistic tests, $\mathcal{SCC}$, and $\mathcal{SC}$. ■

Proof of Thm. 8.19. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\text{MTe,ete}} s_2$, i.e., assume that for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $t \in \mathbb{R}_{\geq 0}$:

$$prob(\mathcal{SC}_{\leq t}(s_1, o)) \; = \; prob(\mathcal{SC}_{\leq t}(s_2, o))$$

By virtue of Lemma 8.17, the assumption is equivalent to having that for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$prob(\mathcal{SCC}_{\leq t}(s_1, o, \alpha)) \; = \; prob(\mathcal{SCC}_{\leq t}(s_2, o, \alpha))$$

Due to [22, 16], this is equivalent to having that for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$prob_{\text{d,ete}}(\mathcal{SCC}(s_1, o, \alpha), t) \; = \; prob_{\text{d,ete}}(\mathcal{SCC}(s_2, o, \alpha), t)$$

where $prob_{\text{d,ete}}$ has been defined in the proof of Thm. 8.11.

Let $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ be the $\mathbb{R}_{\geq 0}$-observation system corresponding to an arbitrary fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and consider the interaction system $\mathcal{I}^{\text{RM}}(\mathcal{U}, \mathcal{O})$. Since for all $s \in S$, $\alpha \in A^*$, and $t \in \mathbb{R}_{\geq 0}$ it holds that:

$$prob_{\text{d,ete}}(\mathcal{SCC}(s, o, \alpha), t) \; = \; \begin{cases} \sum\limits_{(s', o') \in S \times O} \frac{\mathcal{D}_{(s,o),a}(s',o')}{\text{E}(s,o)} \cdot \int\limits_0^t \text{E}(s,o) \cdot \text{e}^{-\text{E}(s,o) \cdot x} \cdot prob_{\text{d,ete}}(\mathcal{SCC}(s', o', \alpha'), t - x) \, \text{d}x & \\ \qquad \text{if } \alpha = a \circ \alpha' \text{ and } \text{E}(s,o) > 0 & \\ 1 \qquad \text{if } \alpha = \varepsilon \text{ and } o = \omega & \\ 0 \qquad \text{if } \alpha \neq \varepsilon \text{ and } \text{E}(s,o) = 0 & \\ \qquad \text{or } \alpha = \varepsilon \text{ and } o \neq \omega & \end{cases}$$

and hence:

$$prob_{\text{d,ete}}(\mathcal{SCC}(s, o, \alpha), t) \; = \; \mathcal{M}^{\text{RM},\mathcal{O}}_{\text{ete}}((s, o), \alpha, \mathcal{S}^{\text{RM}}(\mathcal{U}, \mathcal{O}))(t)$$

we immediately derive that the assumption is equivalent to having that for every $\mathbb{R}_{\geq 0}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$\mathcal{M}^{\text{RM},\mathcal{O}}_{\text{ete}}((s_1, o), \alpha, \mathcal{S}^{\text{RM}}(\mathcal{U}, \mathcal{O}))(t) \; = \; \mathcal{M}^{\text{RM},\mathcal{O}}_{\text{ete}}((s_2, o), \alpha, \mathcal{S}^{\text{RM}}(\mathcal{U}, \mathcal{O}))(t)$$

which means that $s_1 \sim_{\text{Te},\mathcal{M}^{\text{RM}}_{\text{ete}}} s_2$. ■

Proof of Thm. 8.20. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\text{MT,sbs}} s_2$, i.e., assume that for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob(\mathcal{SC}_{\leq\theta}(s_1, o)) \; = \; prob(\mathcal{SC}_{\leq\theta}(s_2, o))$$

By virtue of Lemma 8.18, the assumption is equivalent to having that for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob(\mathcal{SCC}_{\leq\theta}(s_1, o, \alpha)) \; = \; prob(\mathcal{SCC}_{\leq\theta}(s_2, o, \alpha))$$

Due to [16], this is equivalent to having that for every fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob_{\text{d,sbs}}(\mathcal{SCC}(s_1, o, \alpha), \theta) \; = \; prob_{\text{d,sbs}}(\mathcal{SCC}(s_2, o, \alpha), \theta)$$

where $prob_{\text{d,sbs}}$ has been defined in the proof of Thm. 8.12.

Let $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ be the $\mathbb{R}_{\geq 0}$-observation system corresponding to an arbitrary fully stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and consider the interaction system $\mathcal{I}^{\text{RM}}(\mathcal{U}, \mathcal{O})$. Since for all $s \in S$, $\alpha \in A^*$, and $\theta \in (\mathbb{R}_{\geq 0})^*$ it holds that:

$$prob_{\text{d,sbs}}(\mathcal{SCC}(s, o, \alpha), \theta) \; = \; \begin{cases} \sum\limits_{(s', o') \in S \times O} \frac{\mathcal{D}_{(s,o),a}(s',o')}{\text{E}(s,o)} \cdot (1 - \text{e}^{-\text{E}(s,o) \cdot t}) \cdot prob_{\text{d,sbs}}(\mathcal{SCC}(s', o', \alpha'), \theta') & \\ \qquad \text{if } \alpha = a \circ \alpha' \text{ and } \theta = t \circ \theta' \text{ and } \text{E}(s,o) > 0 & \\ 1 \qquad \text{if } \alpha = \varepsilon \text{ and } o = \omega & \\ 0 \qquad \text{if } \alpha \neq \varepsilon \text{ and } \theta = \varepsilon & \\ \qquad \text{or } \alpha \neq \varepsilon \text{ and } \theta \neq \varepsilon \text{ and } \text{E}(s,o) = 0 & \\ \qquad \text{or } \alpha = \varepsilon \text{ and } o \neq \omega & \end{cases}$$

and hence:

$$prob(\mathcal{SCC}_{\text{d,sbs}}(s, o, \alpha), \theta) \; = \; \mathcal{M}^{\text{RM},\mathcal{O}}_{\text{sbs}}((s, o), \alpha, \mathcal{S}^{\text{RM}}(\mathcal{U}, \mathcal{O}))(\theta)$$

we immediately derive that the assumption is equivalent to having that for every $\mathbb{R}_{\geq 0}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$\mathcal{M}_{\rm sbs}^{\rm RM,\mathcal{O}}((s_1,o),\alpha,\mathcal{S}^{\rm RM}(\mathcal{U},\mathcal{O}))(\theta) \;=\; \mathcal{M}_{\rm sbs}^{\rm RM,\mathcal{O}}((s_2,o),\alpha,\mathcal{S}^{\rm RM}(\mathcal{U},\mathcal{O}))(\theta)$$

which means that $s_1 \sim_{\rm Te,\mathcal{M}_{\rm sbs}^{\rm RM}} s_2$. ■

Proof of Lemma 9.4. The proof is identical to that of Lemma 8.4 up to summations over $a \in A$ (which are no longer present) and the use of conditional exit rates in place of exit rates. ■

Proof of Lemma 9.5. The proof is identical to that of Lemma 8.5 up to summations over $a \in A$ (which are no longer present) and the use of conditional exit rates and $\mathcal{M}_{\rm ete,R}$ in place of exit rates and $\mathcal{M}_{\rm ete}$. ■

Proof of Lemma 9.6. The proof is identical to that of Lemma 8.6 up to summations over $a \in A$ (which are no longer present) and the use of conditional exit rates and $\mathcal{M}_{\rm sbs,R}$ in place of exit rates and $\mathcal{M}_{\rm sbs}$. ■

Proof of Thm. 9.7. The proof is identical to that of Thm. 8.7 up to the use of conditional exit rates, $\mathcal{M}_{\rm ete,R}$, Lemma 9.4, and Lemma 9.5 in place of exit rates, $\mathcal{M}_{\rm ete}$, Lemma 8.4, and Lemma 8.5.

Proof of Thm. 9.8. The proof is identical to that of Thm. 8.8 up to the use of conditional exit rates, $\mathcal{M}_{\rm sbs,R}$, Lemma 9.4, and Lemma 9.6 in place of exit rates, $\mathcal{M}_{\rm sbs}$, Lemma 8.4, and Lemma 8.6. ■

Proof of Thm. 9.11. The proof is identical to that of Thm. 8.11 up to the use of conditional exit rates, $prob_{\rm R}$, $\sim_{\rm MTr,ete,R}$, and $\mathcal{M}_{\rm ete,R}$ in place of exit rates, $prob$, $\sim_{\rm MTr,ete}$, and $\mathcal{M}_{\rm ete}$. ■

Proof of Thm. 9.12. The proof is identical to that of Thm. 8.12 up to the use of conditional exit rates, $prob_{\rm R}$, $\sim_{\rm MTr,sbs,R}$, and $\mathcal{M}_{\rm sbs,R}$ in place of exit rates, $prob$, $\sim_{\rm MTr,sbs}$, and $\mathcal{M}_{\rm sbs}$. ■

Proof of Lemma 9.17. The proof is identical to that of Lemma 6.8 up to the use of $\sim_{\rm MTe,ete,R}$, reactive stochastic tests, $prob_{\rm R}$, and $\mathcal{SCC}_{\le t,\rm R}$ in place of $\sim_{\rm PTe}$, reactive probabilistic tests, $prob$, and $\mathcal{SCC}$. ■

Proof of Lemma 9.18. The proof is identical to that of Lemma 6.8 up to the use of $\sim_{\rm MTe,sbs,R}$, reactive stochastic tests, $prob_{\rm R}$, and $\mathcal{SCC}_{\le \theta,\rm R}$ in place of $\sim_{\rm PTe}$, reactive probabilistic tests, $prob$, and $\mathcal{SCC}$. ■

Proof of Thm. 9.19. The proof is identical to that of Thm. 8.19 up to the use of $\sim_{\rm MTe,ete,R}$, reactive stochastic tests, $prob_{\rm R}$, $\mathcal{SC}_{\le t,\rm R}$, Lemma 9.17, $\mathcal{SCC}_{\le t,\rm R}$, conditional exit rates, and $\mathcal{M}_{\rm ete,R}^{\rm RM}$ in place of $\sim_{\rm MTe,ete}$, fully stochastic tests, $prob$, $\mathcal{SC}_{\le t}$, Lemma 8.17, $\mathcal{SCC}_{\le t}$, exit rates, and $\mathcal{M}_{\rm ete}^{\rm RM}$. ■

Proof of Thm. 9.20. The proof is identical to that of Thm. 8.20 up to the use of $\sim_{\rm MTe,sbs,R}$, reactive stochastic tests, $prob_{\rm R}$, $\mathcal{SC}_{\le \theta,\rm R}$, Lemma 9.18, $\mathcal{SCC}_{\le \theta,\rm R}$, conditional exit rates, and $\mathcal{M}_{\rm sbs,R}^{\rm RM}$ in place of $\sim_{\rm MTe,sbs}$, fully stochastic tests, $prob$, $\mathcal{SC}_{\le \theta}$, Lemma 8.18, $\mathcal{SCC}_{\le \theta}$, exit rates, and $\mathcal{M}_{\rm sbs}^{\rm RM}$. ■

Proof of Thm. 10.4. Let $s_1, s_2 \in S$. Assume that $s_1 \sim_{\rm MB,N} s_2$ due to some Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a} \mathcal{D}_1$ implies $s'_2 \xrightarrow{a} \mathcal{D}_2$ with $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$ and $\mathcal{D}_1(S) = \mathcal{D}_2(S)$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\begin{array}{rcl}
\bigcup\limits_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{(\mathcal{D}_1(\bigcup \mathcal{G}), \mathcal{D}_1(S))\} & \subseteq & \bigcup\limits_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{(\mathcal{D}_2(\bigcup \mathcal{G}), \mathcal{D}_2(S))\} \\
\bigcup\limits_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{(\mathcal{D}_2(\bigcup \mathcal{G}), \mathcal{D}_2(S))\} & \subseteq & \bigcup\limits_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{(\mathcal{D}_1(\bigcup \mathcal{G}), \mathcal{D}_1(S))\}
\end{array}$$

or equivalently:

$$\bigcup\limits_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{(\mathcal{D}_1(\bigcup \mathcal{G}), \mathcal{D}_1(S))\} \;=\; \bigcup\limits_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{(\mathcal{D}_2(\bigcup \mathcal{G}), \mathcal{D}_2(S))\}$$

Since for all $s \in S$, $a \in A$, $\mathcal{G} \in 2^{S/\mathcal{B}}$, and $t \in \mathbb{R}_{\ge 0}$ it holds that $\mathcal{M}_{\rm ete,N}(s, a, \bigcup \mathcal{G})(t) = \{0\}$ when s has no a-transition, otherwise:

$$\begin{array}{rcl}
\mathcal{M}_{\rm ete,N}(s, a, \bigcup \mathcal{G})(t) & = & \bigcup\limits_{s \xrightarrow{a} \mathcal{D}} \{\int\limits_0^t \mathcal{D}(S) \cdot {\rm e}^{-\mathcal{D}(S) \cdot x} \cdot \sum\limits_{s' \in \bigcup \mathcal{G}} \frac{\mathcal{D}(s')}{\mathcal{D}(S)} \, {\rm d}x\} \\
& = & \bigcup\limits_{s \xrightarrow{a} \mathcal{D}} \{(\int\limits_0^t \mathcal{D}(S) \cdot {\rm e}^{-\mathcal{D}(S) \cdot x} \, {\rm d}x) \cdot \frac{1}{\mathcal{D}(S)} \cdot \sum\limits_{s' \in \bigcup \mathcal{G}} \mathcal{D}(s')\} \\
& = & \bigcup\limits_{s \xrightarrow{a} \mathcal{D}} \{\frac{1 - {\rm e}^{-\mathcal{D}(S) \cdot t}}{\mathcal{D}(S)} \cdot \mathcal{D}(\bigcup \mathcal{G})\}
\end{array}$$

we derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_{\mathrm{ete,N}}(s'_1, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}_{\mathrm{ete,N}}(s'_2, a, \bigcup \mathcal{G})$$

which means that $\mathcal{B}$ is an $\mathcal{M}_{\mathrm{ete,N}}$-bisimulation such that $(s_1, s_2) \in \mathcal{B}$. In other words, $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,N}}} s_2$. ■

Proof of Thm. 10.5. Let $s_1, s_2 \in S$. Assume that $s_1 \sim_{\mathrm{MB,N}} s_2$ due to some Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a} \mathcal{D}_1$ implies $s'_2 \xrightarrow{a} \mathcal{D}_2$ with $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$ and $\mathcal{D}_1(S) = \mathcal{D}_2(S)$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\bigcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{(\mathcal{D}_1(\bigcup \mathcal{G}), \mathcal{D}_1(S))\} \subseteq \bigcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{(\mathcal{D}_2(\bigcup \mathcal{G}), \mathcal{D}_2(S))\}$$

$$\bigcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{(\mathcal{D}_2(\bigcup \mathcal{G}), \mathcal{D}_2(S))\} \subseteq \bigcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{(\mathcal{D}_1(\bigcup \mathcal{G}), \mathcal{D}_1(S))\}$$

or equivalently:

$$\bigcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \{(\mathcal{D}_1(\bigcup \mathcal{G}), \mathcal{D}_1(S))\} = \bigcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \{(\mathcal{D}_2(\bigcup \mathcal{G}), \mathcal{D}_2(S))\}$$

Since for all $s \in S$, $a \in A$, $\mathcal{G} \in 2^{S/\mathcal{B}}$, and $\theta \in (\mathbb{R}_{\geq 0})^*$ it holds that $\mathcal{M}_{\mathrm{ete,N}}(s, a, \bigcup \mathcal{G})(\theta) = \{0\}$ when $\theta = \varepsilon$ or s has no a-transition, otherwise for $\theta = t \circ \theta'$:

$$\begin{aligned}
\mathcal{M}_{\mathrm{sbs,N}}(s, a, \bigcup \mathcal{G})(\theta) &= \bigcup_{s \xrightarrow{a} \mathcal{D}} \{(1 - \mathrm{e}^{-\mathcal{D}(S) \cdot t}) \cdot \sum_{s' \in \bigcup \mathcal{G}} \tfrac{\mathcal{D}(s')}{\mathcal{D}(S)}\} \\
&= \bigcup_{s \xrightarrow{a} \mathcal{D}} \{\tfrac{1 - \mathrm{e}^{-\mathcal{D}(S) \cdot t}}{\mathcal{D}(S)} \cdot \sum_{s' \in \bigcup \mathcal{G}} \mathcal{D}(s')\} \\
&= \bigcup_{s \xrightarrow{a} \mathcal{D}} \{\tfrac{1 - \mathrm{e}^{-\mathcal{D}(S) \cdot t}}{\mathcal{D}(S)} \cdot \mathcal{D}(\bigcup \mathcal{G})\}
\end{aligned}$$

we derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_{\mathrm{sbs,N}}(s'_1, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}_{\mathrm{sbs,N}}(s'_2, a, \bigcup \mathcal{G})$$

which means that $\mathcal{B}$ is an $\mathcal{M}_{\mathrm{sbs,N}}$-bisimulation such that $(s_1, s_2) \in \mathcal{B}$. In other words, $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,N}}} s_2$. ■

Proof of Thm. 10.8. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{MTr,ete,N}} s_2$, i.e., assume that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that:
$$\mathit{prob}(\mathcal{CC}_{\leq t}(z_{s_1}, \alpha)) \; = \; \mathit{prob}(\mathcal{CC}_{\leq t}(z_{s_2}, \alpha))$$

- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ such that:
$$\mathit{prob}(\mathcal{CC}_{\leq t}(z_{s_2}, \alpha)) \; = \; \mathit{prob}(\mathcal{CC}_{\leq t}(z_{s_1}, \alpha))$$

Due to [22, 16], this is equivalent to having that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that:
$$\mathit{prob}_{\mathrm{d,ete}}(\mathcal{CC}(z_{s_1}, \alpha), t) \; = \; \mathit{prob}_{\mathrm{d,ete}}(\mathcal{CC}(z_{s_2}, \alpha), t)$$

- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ such that:
$$\mathit{prob}_{\mathrm{d,ete}}(\mathcal{CC}(z_{s_2}, \alpha), t) \; = \; \mathit{prob}_{\mathrm{d,ete}}(\mathcal{CC}(z_{s_1}, \alpha), t)$$

where $\mathit{prob}_{\mathrm{d,ete}}$ has been defined in the proof of Thm. 8.11.

Since for all $s \in S$, $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in \mathit{Res}(s)$, $\alpha \in A^*$, and $t \in \mathbb{R}_{\geq 0}$ it holds that:

$$\mathit{prob}_{\mathrm{d,ete}}(\mathcal{CC}(z_s, \alpha), t) \; = \; \begin{cases} \sum\limits_{z_{s'} \in Z} \frac{\mathcal{D}(z_{s'})}{\mathcal{D}(Z)} \cdot \int\limits_0^t \mathcal{D}(Z) \cdot \mathrm{e}^{-\mathcal{D}(Z) \cdot x} \cdot \mathit{prob}_{\mathrm{d,ete}}(\mathcal{CC}(z_{s'}, \alpha'), t - x) \, \mathrm{d}x & \\ & \text{if } \alpha = a \circ \alpha' \text{ and } \exists \mathcal{D} \in [Z \to \mathbb{R}_{\geq 0}].\, z_s \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} \\ 1 & \text{if } \alpha = \varepsilon \\ 0 & \text{if } \alpha = a \circ \alpha' \text{ and } \nexists \mathcal{D} \in [Z \to \mathbb{R}_{\geq 0}].\, z_s \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} \end{cases}$$

and hence:

$$prob_{\rm d,ete}(\mathcal{CC}(z_s, \alpha), t) \; \in \; \mathcal{M}_{\rm ete,N}(s, \alpha, S)(t)$$

we immediately derive that the assumption is equivalent to having that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$\mathcal{M}_{\rm ete,N}(s_1, \alpha, S)(t) \; = \; \mathcal{M}_{\rm ete,N}(s_2, \alpha, S)(t)$$

which means that $s_1 \sim_{\rm Tr, \mathcal{M}_{ete,N}} s_2$. ■

Proof of Thm. 10.9. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\rm MTr,sbs,N} s_2$, i.e., assume that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that:
$$\mathit{prob}(\mathcal{CC}_{\leq \theta}(z_{s_1}, \alpha)) \; = \; \mathit{prob}(\mathcal{CC}_{\leq \theta}(z_{s_2}, \alpha))$$
- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ such that:
$$\mathit{prob}(\mathcal{CC}_{\leq \theta}(z_{s_2}, \alpha)) \; = \; \mathit{prob}(\mathcal{CC}_{\leq \theta}(z_{s_1}, \alpha))$$

Due to [16], this is equivalent to having that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that:
$$prob_{\rm d,sbs}(\mathcal{CC}(z_{s_1}, \alpha), \theta) \; = \; prob_{\rm d,sbs}(\mathcal{CC}(z_{s_2}, \alpha), \theta)$$
- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ such that:
$$prob_{\rm d,sbs}(\mathcal{CC}(z_{s_2}, \alpha), \theta) \; = \; prob_{\rm d,sbs}(\mathcal{CC}(z_{s_1}, \alpha), \theta)$$

where $prob_{\rm d,sbs}$ has been defined in the proof of Thm. 8.12.
Since for all $s \in S$, $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in \mathit{Res}(s)$, $\alpha \in A^*$, and $\theta \in (\mathbb{R}_{\geq 0})^*$ it holds that:

$$prob_{\rm d,sbs}(\mathcal{CC}(z_s, \alpha), \theta) \; = \; \begin{cases} \sum\limits_{z_{s'} \in Z} \frac{\mathcal{D}(z_{s'})}{\mathcal{D}(Z)} \cdot (1 - {\rm e}^{-\mathcal{D}(Z) \cdot t}) \cdot prob_{\rm d,sbs}(\mathcal{CC}(z_{s'}, \alpha'), \theta') & \\ & \text{if } \alpha = a \circ \alpha' \text{ and } \theta = t \circ \theta' \text{ and } \exists \mathcal{D} \in [Z \rightarrow \mathbb{R}_{\geq 0}].\, z_s \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} \\ 1 & \text{if } \alpha = \varepsilon \\ 0 & \text{if } \alpha \neq \varepsilon \text{ and } \theta = \varepsilon \\ & \text{or } \alpha = a \circ \alpha' \text{ and } \theta \neq \varepsilon \text{ and } \nexists \mathcal{D} \in [Z \rightarrow \mathbb{R}_{\geq 0}].\, z_s \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} \end{cases}$$

and hence:

$$prob_{\rm d,sbs}(\mathcal{CC}(z_s, \alpha), \theta) \; \in \; \mathcal{M}_{\rm sbs,N}(s, \alpha, S)(\theta)$$

we immediately derive that the assumption is equivalent to having that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$\mathcal{M}_{\rm sbs,N}(s_1, \alpha, S)(\theta) \; = \; \mathcal{M}_{\rm sbs,N}(s_2, \alpha, S)(\theta)$$

which means that $s_1 \sim_{\rm Tr, \mathcal{M}_{sbs,N}} s_2$. ■

Proof of Thm. 10.14. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\rm MTe,ete,N} s_2$, i.e., assume that for every nondeterministic and stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1, o)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2, o)$ such that:
$$\mathit{prob}(\mathcal{SCC}_{\leq t}(z_{s_1,o}, \alpha)) \; = \; \mathit{prob}(\mathcal{SCC}_{\leq t}(z_{s_2,o}, \alpha))$$
- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2, o)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1, o)$ such that:
$$\mathit{prob}(\mathcal{SCC}_{\leq t}(z_{s_2,o}, \alpha)) \; = \; \mathit{prob}(\mathcal{SCC}_{\leq t}(z_{s_1,o}, \alpha))$$

Due to [22, 16], this is equivalent to having that for every nondeterministic and stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1, o)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2, o)$ such that:
$$prob_{\rm d,ete}(\mathcal{SCC}(z_{s_1,o}, \alpha), t) \; = \; prob_{\rm d,ete}(\mathcal{SCC}(z_{s_2,o}, \alpha), t)$$
- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2, o)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1, o)$ such that:
$$prob_{\rm d,ete}(\mathcal{SCC}(z_{s_2,o}, \alpha), t) \; = \; prob_{\rm d,ete}(\mathcal{SCC}(z_{s_1,o}, \alpha), t)$$

where $prob_{\mathrm{d,ete}}$ has been defined in the proof of Thm. 8.11.

Let $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ be the $\mathbb{R}_{\geq 0}$-observation system corresponding to an arbitrary nondeterministic and stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and consider the interaction system $\mathcal{I}^{\mathrm{RM}}(\mathcal{U}, \mathcal{O})$ where $\mathcal{U}$ is the NMLTS under examination. Since for all $s \in S$, $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s, o)$, $\alpha \in A^*$, and $t \in \mathbb{R}_{\geq 0}$ it holds that:

$$prob_{\mathrm{d,ete}}(\mathcal{SCC}(z_{s,o}, \alpha), t) \; = \; \begin{cases} \sum\limits_{z_{s',o'} \in Z} \frac{\mathcal{D}(z_{s',o'})}{\mathcal{D}(Z)} \cdot \int\limits_0^t \mathcal{D}(Z) \cdot \mathrm{e}^{-\mathcal{D}(Z) \cdot x} \cdot prob_{\mathrm{d,ete}}(\mathcal{SCC}(z_{s',o'}, \alpha'), t - x) \, \mathrm{d}x & \\ \quad \text{if } \alpha = a \circ \alpha' \text{ and } \exists \mathcal{D} \in [Z \to \mathbb{R}_{\geq 0}].\, z_{s,o} \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} & \\ 1 \quad \text{if } \alpha = \varepsilon \text{ and } o = \omega & \\ 0 \quad \text{if } \alpha = a \circ \alpha' \text{ and } \nexists \mathcal{D} \in [Z \to \mathbb{R}_{\geq 0}].\, z_{s,o} \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} & \\ \quad \text{or } \alpha = \varepsilon \text{ and } o \neq \omega & \end{cases}$$

and hence:

$$prob_{\mathrm{d,ete}}(\mathcal{SCC}(z_{s,o}, \alpha), t) \; \in \; \mathcal{M}^{\mathrm{RM},\mathcal{O}}_{\mathrm{ete,N}}((s, o), \alpha, \mathcal{S}^{\mathrm{RM}}(\mathcal{U}, \mathcal{O}))(t)$$

we immediately derive that the assumption is equivalent to having that for every $\mathbb{R}_{\geq 0}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$\mathcal{M}^{\mathrm{RM},\mathcal{O}}_{\mathrm{ete,N}}((s_1, o), \alpha, \mathcal{S}^{\mathrm{RM}}(\mathcal{U}, \mathcal{O}))(t) \; = \; \mathcal{M}^{\mathrm{RM},\mathcal{O}}_{\mathrm{ete,N}}((s_2, o), \alpha, \mathcal{S}^{\mathrm{RM}}(\mathcal{U}, \mathcal{O}))(t)$$

which means that $s_1 \sim_{\mathrm{Te}, \mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,N}}} s_2$. ■

Proof of Thm. 10.15. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{MTe,sbs,N}} s_2$, i.e., assume that for every nondeterministic and stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

- For each $\mathcal{Z}_1 \in Res(s_1, o)$ there exists $\mathcal{Z}_2 \in Res(s_2, o)$ such that:
$$prob(\mathcal{SCC}_{\leq \theta}(z_{s_1,o}, \alpha)) \; = \; prob(\mathcal{SCC}_{\leq \theta}(z_{s_2,o}, \alpha))$$
- For each $\mathcal{Z}_2 \in Res(s_2, o)$ there exists $\mathcal{Z}_1 \in Res(s_1, o)$ such that:
$$prob(\mathcal{SCC}_{\leq \theta}(z_{s_2,o}, \alpha)) \; = \; prob(\mathcal{SCC}_{\leq \theta}(z_{s_1,o}, \alpha))$$

Due to [16], this is equivalent to having that for every nondeterministic and stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

- For each $\mathcal{Z}_1 \in Res(s_1, o)$ there exists $\mathcal{Z}_2 \in Res(s_2, o)$ such that:
$$prob_{\mathrm{d,sbs}}(\mathcal{SCC}(z_{s_1,o}, \alpha), \theta) \; = \; prob_{\mathrm{d,sbs}}(\mathcal{SCC}(z_{s_2,o}, \alpha), \theta)$$
- For each $\mathcal{Z}_2 \in Res(s_2, o)$ there exists $\mathcal{Z}_1 \in Res(s_1, o)$ such that:
$$prob_{\mathrm{d,sbs}}(\mathcal{SCC}(z_{s_2,o}, \alpha), \theta) \; = \; prob_{\mathrm{d,sbs}}(\mathcal{SCC}(z_{s_1,o}, \alpha), \theta)$$

where $prob_{\mathrm{d,sbs}}$ has been defined in the proof of Thm. 8.12.

Let $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ be the $\mathbb{R}_{\geq 0}$-observation system corresponding to an arbitrary nondeterministic and stochastic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and consider the interaction system $\mathcal{I}^{\mathrm{RM}}(\mathcal{U}, \mathcal{O})$ where $\mathcal{U}$ is the NMLTS under examination. Since for all $s \in S$, $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s, o)$, $\alpha \in A^*$, and $\theta \in (\mathbb{R}_{\geq 0})^*$ it holds that:

$$prob_{\mathrm{d,sbs}}(\mathcal{SCC}(z_{s,o}, \alpha), \theta) \; = \; \begin{cases} \sum\limits_{z_{s',o'} \in Z} \frac{\mathcal{D}(z_{s',o'})}{\mathcal{D}(Z)} \cdot (1 - \mathrm{e}^{-\mathcal{D}(Z) \cdot t}) \cdot prob_{\mathrm{d,sbs}}(\mathcal{SCC}(z_{s',o'}, \alpha'), \theta') & \\ \quad \text{if } \alpha = a \circ \alpha' \text{ and } \theta = t \circ \theta' \text{ and } \exists \mathcal{D} \in [Z \to \mathbb{R}_{\geq 0}].\, z_{s,o} \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} & \\ 1 \quad \text{if } \alpha = \varepsilon \text{ and } o = \omega & \\ 0 \quad \text{if } \alpha \neq \varepsilon \text{ and } \theta = \varepsilon & \\ \quad \text{or } \alpha = a \circ \alpha' \text{ and } \theta \neq \varepsilon \text{ and } \nexists \mathcal{D} \in [Z \to \mathbb{R}_{\geq 0}].\, z_{s,o} \xrightarrow{a}_{\mathcal{Z}} \mathcal{D} & \\ \quad \text{or } \alpha = \varepsilon \text{ and } o \neq \omega & \end{cases}$$

and hence:

$$prob_{\mathrm{d,sbs}}(\mathcal{SCC}(z_{s,o}, \alpha), \theta) \; \in \; \mathcal{M}^{\mathrm{RM},\mathcal{O}}_{\mathrm{sbs,N}}((s, o), \alpha, \mathcal{S}^{\mathrm{RM}}(\mathcal{U}, \mathcal{O}))(\theta)$$

we immediately derive that the assumption is equivalent to having that for every $\mathbb{R}_{\geq 0}$-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$\mathcal{M}^{\mathrm{RM},\mathcal{O}}_{\mathrm{sbs,N}}((s_1, o), \alpha, \mathcal{S}^{\mathrm{RM}}(\mathcal{U}, \mathcal{O}))(\theta) \; = \; \mathcal{M}^{\mathrm{RM},\mathcal{O}}_{\mathrm{sbs,N}}((s_2, o), \alpha, \mathcal{S}^{\mathrm{RM}}(\mathcal{U}, \mathcal{O}))(\theta)$$

which means that $s_1 \sim_{\mathrm{Te}, \mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,N}}} s_2$. ■

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