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INTERVAL BOUNDS FOR THE OPTIMAL BURN-IN TIMES FOR CONCAVE OR CONVEX REWARD FUNCTIONS

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Abstract

An interesting problem in reliability is to determine the optimal burn-in time. In a previous work, the authors studied the solution of such a problem under a particular cost structure. It has been shown there that a key role in the problem is played by a function ρ, representing the reward coming from the use of a component in the field. A relevant case in this investigation is the one when ρ is linear.

In this paper, we explore further the linear case and use its solutions as a benchmark for determining the locally optimal times when the function ρ is not linear or under a different cost structure.

Keywords: Burn-in; Bathtub shape; Multiple change points distributions; Reward functions.

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Introduction

A well-known problem in reliability is determining the optimal duration of burn-in for a component to be put into operations. This topic is very widespread; we will recall, in the present section and in the next one, basic definitions and results about it and essential references.

In the present paper, we take the cue from [3] and broaden some more analytical aspects of the optimization problem related to the burn-in time (see in particular Theorem 2.1 and Corollary 2.1 below). In [3], the optimal burn-in time problem has been analyzed from the point of view of its connections with ageing and risk-aversion concepts. Such an optimization problem has been formulated under a particular reward

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function R_ρ, defined by

$$R_\rho(t,\delta) = [\rho(t-\delta) + \tilde{k}]\mathbf{1}_{\{t\geq\delta\}} + k\mathbf{1}_{\{t<\delta\}} + c\min(t,\delta). \quad (0.1)$$

We investigate here in detail the role of the term ρ within the reward function in Eq. (0.1). Also, we will show that this particular reward function may be used as a reference point for the solution of the optimization problem concerning different reward functions (see Section 5).
In this respect, we recall that several cost structures have been proposed in the literature (for a detailed survey, see [1] and references therein) and several are the versions of the problem we could obtain by modifying our reward function, so as to model different kinds of usage of burned-in units.

The reward function $R_\rho(t,\delta)$ depends on

- the burn-in time, $\delta > 0$;
- the total lifetime of the unit, $t > 0$ (burn-in time plus time in operations).

In the following, t will represent the value taken by a random variable T, with probability distribution G, and δ will be the quantity to be determined in view of optimizing the expected value of $R_\rho(T,\delta)$.

The reward function in Eq. (0.1) consists of:

- a cost of conducting burn-in, $c\min(t,\delta)$, proportional to the actual burn-in time, by means of the constant cost rate $c < 0$;
- a fixed cost due to failure during the burn-in, $k < 0$;
- a fixed cost due to failure during the use, $\tilde{k} < 0$;
- a reward $\rho : [0,+\infty) \to [0,+\infty)$, that is an increasing function of the duration of the unit in the operations.

In order to avoid the optimization problem to be trivial, we assume $\tilde{k} < k < 0$, i.e. the cost of losing the unit during burn-in is smaller than the cost of failure during operation.

Even if the case when ρ is linear was discussed at length in [3], some analytical and technical aspects require a further investigation, that we will carry on here. Subsequently, part of the present paper is devoted to the application of our results

for the linear case to the case when ρ is not linear. In fact, the solution for the burn-in problem in the non-linear case does not follow straightly from the results we got in the linear case. Finding a solution for some relevant non-linear cases, by providing a generalization of the results for the linear case, is the central aim of this paper.
As a main achievement in this respect, we will provide upper and lower bounds for the optimal burn-in time, when ρ is concave or convex. We can determine the interval, or more generally the union of intervals $\mathcal{S}$, that will contain the optimal burn-in time, by setting up a comparison among different functions ρ. In providing these bounds, a key role is played by the results about the optimal burn-in time δ_l^* obtained in the case when ρ is linear. For the computation of such a δ_l^*, we can rely on the (sufficient and necessary) conditions provided in [3, Prop. 2]. Furthermore, we give conditions on δ_l^* under which the set $\mathcal{S}$ is bounded.

The paper is organized as follows. In Section 1, we recall definitions and results from [3], that are preliminary for our developments here. In Section 2, we study in detail the burn-in problem associated with a reward function of the kind in Eq. (0.1). We provide analytical-type results for the case when ρ is linear, releasing the hypothesis $\overline{G}$ bathtub or upside down bathtub. We will use these results as a benchmark in Sections 3 and 4. In Section 3, we come back to consider $\overline{G}$ bathtub or upside down bathtub and use results of Section 2 to obtain bounds for the optimal burn-in time in the case when ρ is concave or convex. In Section 4, we extend the results of Section 3, again dropping the hypothesis $\overline{G}$ bathtub or upside down bathtub. Finally, in Section 5, we compare the burn-in problem corresponding to the reward function in Eq. (0.1) to another version of it, obtained by implementing in the model a surcharge for a failure occurred during a mission time. The conditions on reward function needed for applying previous theorems do not hold. We can however provide some bounds for the optimal burn-in time in the case when ρ is concave or convex, starting from the local optimal burn-in times for the linear case.

1. Notation and basic results

Let T be a random variable, representing the lifetime of the unit that will undergo the burn-in, with probability distribution G, survival function $\overline{G}$, and density g. Its hazard rate is denoted by $r(t) = r_{\overline{G}}(t) = \dfrac{g(t)}{\overline{G}(t)}$.
Such a definition implies that $\overline{G}(t) > 0$ for all t where $r(t)$ is defined.
By assuming $\overline{G}$ twice differentiable, we denote by

$$\alpha(t) = \alpha_{\overline{G}}(t) = -\frac{g'(t)}{g(t)} \tag{1.1}$$

the *risk aversion coefficient* associated with $\overline{G}$ (see e.g. [3, 6, 8]).

For our purposes, it is convenient to recall the definition of ageing properties of $\overline{G}$ (or, that is the same, of G) in terms of monotonicity properties of $r(t)$ and some connected concepts:

Definition 1.1. $\sigma \in (0, +\infty)$ is called a *change* (or *turning*) *point* for $r(t)$ (or for G), if, for some $\varepsilon \in (0, \sigma)$, it is such that $r(t)$ is decreasing (increasing) for $t \in (\sigma - \varepsilon, \sigma)$ and increasing (decreasing) for $t \in (\sigma, \sigma + \varepsilon)$.

Remark 1.1. Since $\overline{G}$ twice differentiable implies r differentiable, we can equivalently define σ as a change point, if it is a root of r' with odd multiplicity, i.e. a local minimum or maximum point for r.
For a result on change points, also involving their relation with the coefficient $\alpha(t)$, see [6].

Definition 1.2. We say that $\overline{G}$ is

- IFR (DFR), if $r(t)$ is increasing (decreasing) for any $t \geq 0$;
- δ-IFR (δ-DFR), if $r(t)$ is increasing (decreasing) for any $t \geq \delta$;
- bathtub shaped, if an only change point $\sigma \in (0, +\infty)$ exists such that $r(t)$ is decreasing for $t \in (0, \sigma)$ and increasing for $t \in (\sigma, +\infty)$;
- upside down bathtub (= UB) shaped, if an only change point $\sigma \in (0, +\infty)$ exists such that $r(t)$ is increasing for $t \in (0, \sigma)$ and decreasing for $t \in (\sigma, +\infty)$.

From now on, we will focus on the conditions $\overline{G}$ bathtub or upside down bathtub (UB) shaped or with multiple change points. DFR and IFR follow as particular cases of bathtub distributions, for $\sigma = 0$ or $\sigma = +\infty$, while δ-IFR or δ-DFR are not particular cases, in that the monotonicity character of $r(t)$ for $t < \delta$ is not determined.

The criterion under which the optimal burn-in time has to be determined depends on the cost structure of the model:

Definition 1.3. Given a reward function $R_\rho(t, \delta)$, the *optimal burn-in time* is $\delta^*_\rho \in (0, +\infty)$ such that, for any $\delta \in (0, +\infty)$,

$$\mathbb{E}[R_\rho(T, \delta^*_\rho)] \geq \mathbb{E}[R_\rho(T, \delta)].$$

If such an optimal point does not exist, we set $\delta^*_\rho = +\infty$.

In practice, the condition $\delta^*_\rho = +\infty$ will mean than the burn-in will be not carried out.

As mentioned, we will analyze in detail the burn-in problem associated with a reward function of the kind in Eq. (0.1). However the results for this particular case may be used as a benchmark also for different versions of the reward function.

Our optimization problem will amount to maximizing the expected reward

$$\mathcal{R}_\rho(\delta) \equiv \mathbb{E}[R_\rho(T, \delta)] = \tilde{k}\overline{G}(\delta) + kG(\delta) + \int_0^{+\infty} \rho(t)g(t+\delta)dt + c\delta\overline{G}(\delta) + c\int_0^\delta tg(t)dt$$

with respect to the variable δ. We remind that δ has the meaning of the duration of the burn-in procedure.

We use the standard procedure for maximizing a function of one variable: finding δ's such that $\mathcal{R}'_\rho(\delta) = 0$ and looking for the global maximum point among such δ's and δ's where $\mathcal{R}_\rho$ is not differentiable. For sake of simplicity, we will suppose from now on ρ to be twice differentiable. Under our regularity hypotheses on $\overline{G}$ and ρ, the only point where $\mathcal{R}_\rho$ is not differentiable is $\delta = 0$. For $\delta \in (0, +\infty)$,

$$\mathcal{R}'_\rho(\delta) = (k - \tilde{k})g(\delta) + \int_0^{+\infty} \rho(t)g'(t+\delta)dt + c\overline{G}(\delta) \tag{1.2}$$

(see [3]).

2. Locally optimal burn-in times in the linear case

We devote this section to the specific case when ρ linear (i.e. $\rho(t) = \rho_0 t$). Some aspects of this case have already been treated in detail in [3]. Here we complete the study from a more analytical point of view. In the next section, the linear case will be compared with the cases when ρ is, respectively, concave or convex.
The linearity of ρ makes the corresponding optimal burn-in problem more mathematically tractable.

The first good property the linear case manifests, when $\overline{G}$ is bathtub or upside down bathtub, is the uniqueness of the optimal solution δ_l^*. Such a uniqueness result is obtained as a corollary (Corollary 2.1) from Theorem 2.1. Theorem 2.1 is the main result of this section, allowing us also to drop the condition $\overline{G}$ bathtub or upside down bathtub.
Before stating and proving the theorem, we need some preliminary remarks.

First of all, we notice that, when ρ is linear, Eq. (1.2) becomes

$$\begin{aligned} \mathcal{R}_l'(\delta) &= (k-\tilde{k})g(\delta) + \rho_0 \int_0^{+\infty} t g'(t+\delta)dt + c\overline{G}(\delta) && (2.1) \\ &= (k-\tilde{k})g(\delta) + (c-\rho_0)\overline{G}(\delta) && (2.2) \\ &= \overline{G}(\delta)[(k-\tilde{k})r(\delta) + (c-\rho_0)] && (2.3) \end{aligned}$$

(see [3] for detailed computations).

Remark 2.1. By letting

$$b(\delta) := (k-\tilde{k})r(\delta) + (c-\rho_0),$$

we can write

$$\mathcal{R}_l'(\delta) = \overline{G}(\delta)b(\delta);$$

it straightly follows that

$$\operatorname{sgn}(b(\delta)) = \operatorname{sgn}(\mathcal{R}_l'(\delta)),$$

where

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Similarly, since $b'(\delta) = (k - \tilde{k})r'(\delta)$,

$$\text{sgn}(b'(\delta)) = \text{sgn}(r'(\delta)).$$

Lemma 2.1. *If $r'(\delta) \geq 0$ and $\mathcal{R}_l'(\delta) \leq 0$, then $\mathcal{R}_l''(\delta) \geq 0$.*
If $r'(\delta) \leq 0$ and $\mathcal{R}_l'(\delta) \geq 0$, then $\mathcal{R}_l''(\delta) \leq 0$.

Proof. Let $\delta \in (0, +\infty)$ be such that $r'(\delta) \geq 0$ and $\mathcal{R}_l'(\delta) \leq 0$. Then $\mathcal{R}_l''(\delta) \geq 0$. In fact,

$$\mathcal{R}_l''(\delta) = -g(\delta)b(\delta) + \overline{G}(\delta)(k - \tilde{k})r'(\delta);$$

the thesis follows by Remark 2.1 and by noticing that $g(\delta) > 0$ and $\overline{G}(\delta) > 0$. Similarly, if $r'(\delta) \leq 0$ and $\mathcal{R}_l'(\delta) \geq 0$, then $\mathcal{R}_l''(\delta) \leq 0$.

As mentioned in the previous section, we treat the cases when

- $\overline{G}$ is bathtub or upside down bathtub;
- changes its monotonicity an arbitrary finite number n of times.

We point out that the condition $\overline{G}$ is bathtub or upside down bathtub amounts to require that $r'(t)$ has at most one change of sign, while the second condition is equivalent to $r'(t)$ to have n roots with odd multiplicity. This last case is considered in the following Theorem 2.1, allowing us to qualitatively study the graph of the function $\mathcal{R}_l'$ and providing an upper bound for the number of roots of $\mathcal{R}_l'$ and, consequently, for the number of points of local optimum of $\mathcal{R}_l$.

We denote respectively by δ_l^*, $\delta_{min} \in (0, +\infty)$ the point of maximum and of minimum of $\mathcal{R}_l(\delta)$. Analogously to what has been done for the maximum points (see Def. 1.3), we set $\delta_{min} = +\infty$ if the minimum does not exist in $(0, +\infty)$.
Let also $\delta_{fl} \in (0, +\infty)$ be a point of inflection of $\mathcal{R}_l(\delta)$; we can more precisely denote such a point by δ_{fl}^a when it is a point of rising inflection or by δ_{fl}^d if it is a point of falling inflection. Let correspondingly be $\mathcal{D}_{fl}^a$ the set of the points of rising inflection, $\mathcal{D}_{fl}^d$ the one of the points of falling inflection and $\mathcal{D}_{fl} := \mathcal{D}_{fl}^a \cup \mathcal{D}_{fl}^d$.

Theorem 2.1. *Let $\sigma_1 < \cdots < \sigma_n$ be the change points of r and set*
$\sigma_0 = 0,\ \sigma_{n+1} = +\infty$.

- *$\mathcal{R}_l$ admits at most $n + 1$ extremal points in $(0, +\infty)$.*

- *Furthermore,* $\mathcal{D}_{fl} \subseteq \{\sigma_1, \ldots, \sigma_n\}$.

Proof. The proof is structured in three parts. In the first one, we prove that in any open interval (σ_i, σ_{i+1}) falls at most one extremal point, that is one point of maximum or of minimum. For sake of simplicity, we suppose that $r'(t) \neq 0$ for any $t \in \bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1})$, neglecting therefore the occurrence of inflection points for r.
In a second step, we take into account the points $t \in \bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1})$ such that $r'(t) = 0$, showing that the thesis of the theorem is not affected by their existence: also in presence of inflection points for r, we can conclude that $\mathcal{R}_l$ admits no inflection points in $\bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1})$.
In the third part, we take into account the behaviour of r' and $\mathcal{R}_l'$ in the change points.

Notice that, by construction, if $r'(t) > 0$ for some $t \in (\sigma_i, \sigma_{i+1})$, then $r'(t) > 0$ for any $t \in (\sigma_i, \sigma_{i+1})$, i.e. the sign of r' cannot change within any interval (σ_i, σ_{i+1}).

We consider then an arbitrary interval (σ_i, σ_{i+1}). Four alternatives may manifest, by scrutinizing all the possible cases about the signs of r' and $\mathcal{R}_l'(\sigma_i)$:

- $r'(t) > 0$, $\mathcal{R}_l'(\sigma_i) < 0$;
- $r'(t) > 0$, $\mathcal{R}_l'(\sigma_i) > 0$;
- $r'(t) < 0$, $\mathcal{R}_l'(\sigma_i) > 0$;
- $r'(t) < 0$, $\mathcal{R}_l'(\sigma_i) < 0$.

To make the proof more legible, we consider here the strict inequalities, in place of the non-strict ones. As we will see in a second part of the proof, this restriction does not affect the thesis.

Let us discuss the first two cases; the situation presented in the last two cases is symmetric and the proof is analogous.

Thus, let $r'(t) > 0$ and $\mathcal{R}_l'(\sigma_i) < 0$; by continuity, it implies that $\varepsilon > 0$ exists such that $\mathcal{R}_l'(\sigma_i + \varepsilon) < 0$. By Lemma 2.1,

$$\mathcal{R}_l''(\delta) > 0 \ \ \forall\, \delta \in (\sigma_i, \sigma_{i+1}) \text{ or } \delta \in (\sigma_i, \delta_{min}) \subset (\sigma_i, \sigma_{i+1}),$$

if a $\delta_{min} \in (\sigma_i, \sigma_{i+1})$ exists; in this last case, it would be

$$\mathcal{R}_l'(\delta_{min}) = 0 \text{ and } \mathcal{R}_l'(\delta) \geq 0 \ \ \forall\, \delta \in (\delta_{min}, \sigma_{i+1}).$$

In fact, for δ's such that $\mathcal{R}_l'(\delta) > 0$, $\mathcal{R}_l'$'s monotonicity character is not determined by conditions provided by the Lemma, therefore it may both increase (resulting in such a case $\mathcal{R}_l'$ different from zero) and decrease. In this last case, some $\bar{\delta} \in (\delta_{min}, \sigma_{i+1})$ may exist such that $\mathcal{R}_l'(\bar{\delta}) = 0$. However, for $\delta \in (\bar{\delta}, \sigma_{i+1})$, it cannot be $\mathcal{R}_l'(\delta) < 0$, because, by Lemma 2.1, $\mathcal{R}_l''(\bar{\delta}) \geq 0$, so that $\mathcal{R}_l'$ is increasing. Since we cannot a priori exclude the existence of a certain number of $\bar{\delta}$'s, such that $\mathcal{R}_l'(\bar{\delta}) = 0$, internal to intervals $(\delta_{min}, \sigma_{i+1})$, $\mathcal{R}_l'$ could have a priori an infinite number of zeros, corresponding to rising points of inflection for $\mathcal{R}_l$. But, since

$$\mathcal{R}_l'(\delta) > 0 \ \Leftrightarrow \ r(\delta) > \frac{\rho_0 - c}{k - \tilde{k}}$$

and r is increasing, it cannot happen that

$$r(\bar{\delta}) = \frac{\rho_0 - c}{k - \tilde{k}} \text{ for } \bar{\delta} > \delta, \ \bar{\delta} \in (\sigma_i, \sigma_{i+1});$$

therefore no inflection points are admitted in any open interval.
This last argument also covers the case when $r'(t) > 0$ and $\mathcal{R}_l'(\sigma_i) > 0$.

Therefore, in any interval, there is at most one point of maximum or minimum.

Till now, however, we neglect to consider that, for some $t \in (\sigma_i, \sigma_{i+1})$, it could be $r'(t) = 0$. We take into account now this eventuality. Suppose then to be in the case when $r'(t) \geq 0$ and $\mathcal{R}_l'(\sigma_i) > 0$.
The arguments of the previous part of the proof is preserved intact if, for those $\bar{\delta}$'s such that $r'(\bar{\delta}) = 0$, $\mathcal{R}_l'(\bar{\delta}) \neq 0$. By Lemma 2.1, in those points $\mathcal{R}_l'$ is strictly increasing or strictly decreasing. If, on the contrary, for some $\bar{\delta}$ is both $r'(\bar{\delta}) = 0$ and $\mathcal{R}_l'(\bar{\delta}) = 0$, again by Lemma 2.1, $\mathcal{R}_l''(\bar{\delta}) = 0$ as well. This fact however has no repercussions on the proof if $r'(\delta) \neq 0$ for any $\delta \in (\bar{\delta}, \sigma_{i+1})$. If instead $r'(\delta) = 0$ on an entire interval $[\bar{\delta}, \bar{\delta} + \varepsilon) \subseteq [\bar{\delta}, \sigma_{i+1})$, then $\mathcal{R}_l'(\delta) = 0$, for any $\delta \in [\bar{\delta}, \bar{\delta} + \varepsilon)$. We recall that, as described in the previous part of the proof, in $(\sigma_i, \bar{\delta})$ $\mathcal{R}_l'$ is increasing; in $[\bar{\delta} + \varepsilon, \sigma_{i+1})$ its monotonicity character is not determined, but it has necessarily to be $\operatorname{sgn}(\mathcal{R}_l') > 0$. Therefore $[\bar{\delta}, \bar{\delta} + \varepsilon)$ is a set of local minimum points for $\mathcal{R}_l$. However, since $\mathcal{R}_l'(\delta) = 0$ for any $\delta \in [\bar{\delta}, \bar{\delta} + \varepsilon)$, $\mathcal{R}_l(\delta)$ is constant on $[\bar{\delta}, \bar{\delta} + \varepsilon)$. Therefore, actually, the whole interval $[\bar{\delta}, \bar{\delta} + \varepsilon)$ corresponds to a unique minimum value $\mathcal{R}_l(\bar{\delta})$. The same argument holds if $[\bar{\delta}, \bar{\delta} + \varepsilon)$ is a set of local maximum points for $\mathcal{R}_l$.

As concerns the inflection points, since

$$r(\bar{\delta}+\varepsilon) > \frac{\rho_0 - c}{k-\tilde{k}} \quad \text{and} \quad r'(\delta) \geq 0 \text{ for } \delta \in (\bar{\delta}+\varepsilon, \sigma_{i+1}),$$

it must be

$$r(\delta) > \frac{\rho_0 - c}{k-\tilde{k}} \quad \text{for any } \delta \in (\bar{\delta}+\varepsilon, \sigma_{i+1}).$$

Therefore, even if r had inflection points internally to $\bigcup_{i=0}^{n}(\sigma_i, \sigma_{i+1})$, $\mathcal{R}_l$ would have none.

At last, we take into account the behaviour of $\mathcal{R}_l'$ in the change points of r. Such a completion of the analysis is needed since we want to study the optimization problem on the connected set $(0, +\infty)$ and not only on the union of the disjoint open intervals (σ_i, σ_{i+1})'s.
Let us consider the interval $(\sigma_{i-1}, \sigma_{i+1}) = (\sigma_{i-1}, \sigma_i) \cup \{\sigma_i\} \cup (\sigma_i, \sigma_{i+1})$.
As we saw along the proof, for any δ, if $r'(\delta) = 0$ but $\mathcal{R}_l'(\delta) \neq 0$, the arguments we used in the case when $r'(\delta) \neq 0$ are not undermined. We are therefore interested in examining the case when $\mathcal{R}_l'(\sigma_i) = 0$ (being $r'(\sigma_i) = 0$ by definition).
Let us suppose

$$r'(t) \leq 0 \;\text{ for } t \in (\sigma_{i-1}, \sigma_i) \quad \text{and} \quad r'(t) \geq 0 \;\text{ for } t \in (\sigma_i, \sigma_{i+1})$$

(the case $r'(t) \geq 0$ for $t \in (\sigma_{i-1}, \sigma_i)$ and $r'(t) \leq 0$ for $t \in (\sigma_i, \sigma_{i+1})$ is symmetric) and $\mathcal{R}_l'(\sigma_{i-1}) > 0$. As we will see along the proof, this case will also comprehend the illustration of the one when $\mathcal{R}_l'(\sigma_{i-1}) < 0$.
Only the following two subcases are possible:

1. $\delta_l^* \in (\sigma_{i-1}, \sigma_i)$ exists, such that

$$\mathcal{R}_l'(\delta_l^*) = 0 \;\text{ and }\; \mathcal{R}_l'(\delta) < 0 \;\; \forall\, \delta \in (\delta_l^*, \sigma_i);$$

2. $\mathcal{R}_l'(\delta) > 0$ for any $\delta \in (\sigma_{i-1}, \sigma_i)$.

In the first case (and therefore also when $\mathcal{R}_l'(\sigma_{i-1}) < 0$), it has to be
$\mathcal{R}_l'(\sigma_i) \neq 0$. In fact, for any $\bar{\delta} \in (\delta_l^*, \sigma_i)$,

$$\mathcal{R}_l'(\bar{\delta}) < 0 \;\Leftrightarrow\; r(\bar{\delta}) < \frac{\rho_0 - c}{k-\tilde{k}}.$$

On the other hand, $r'(\delta) \leq 0$ for $\delta \in (\sigma_{i-1}, \sigma_i)$ implies that, for any $\delta \in (\bar{\delta}, \sigma_i)$,

$$r(\delta) \leq r(\bar{\delta}) < \frac{\rho_0 - c}{k - \tilde{k}}.$$

Therefore $r(\delta) = \dfrac{\rho_0 - c}{k - \tilde{k}}$ is possible once that is again $r'(\delta) > 0$, that is out of the considered interval (σ_{i-1}, σ_i) and, by continuity, of its closure as well.
Hence $\delta_l^* \in (\sigma_{i-1}, \sigma_i)$ implies $\mathcal{R}_l'(\sigma_i) \neq 0$. Therefore σ_i cannot be a point of minimum (nor of maximum, in the symmetric case).

In the second case, since for $\delta \in (\sigma_i, \sigma_{i+1})$ it has to be $\mathcal{R}_l'(\delta) > 0$, σ_i is a rising (or falling) inflection point.

Therefore only the change points may be inflection points for $\mathcal{R}_l$.

From this theorem, in the particular case $\overline{G}$ is bathtub or UB, one obtains the following

Corollary 2.1. *Let $r'(t)$ have one change of sign. Then $\mathcal{R}_l$ admits at most one point of optimum $\delta_l^* \in (0, +\infty)$.*

An analogous reasoning to the one used in the proof of Theorem 2.1 for showing the non-existence of inflection points leads to the following proposition. It is not a corollary of Theorem 2.1, but it consists in an additional and more precise result we manage in obtaining by requiring stronger hypotheses.

Proposition 2.1. *Let $\overline{G}$ be bathtub shaped and $\mathcal{R}_l'(0) < 0$, then $\delta_l^* = +\infty$.*

Remark 2.2. Notice that we are not questioning here about the global optimum, but about local optima, for which it is enough looking for the points where $\mathcal{R}_\rho'(\delta) = 0$ and $\mathcal{R}_\rho''(\delta) \leq 0$. In particular, we are not considering the possibility that $\delta = 0$ is the optimal burn-in time.
In our treatment of the linear case, however, we need not to know whether δ_l^* is a local or global point of optimum, because, for the use we will make of it in the next section, it is sufficient that δ_l^* is a local optimum.

3. Bounds for the optimal burn-in time for concave and convex ρ

In this section, we come back to consider $\overline{G}$ bathtub or upside down bathtub, but ρ non-linear. In this case, we generally lose the uniqueness of the optimal solution. However, when ρ is concave or convex, we manage at least in providing some bounds for the optimal solution of the corresponding burn-in problem. Such bounds will be obtained by setting up a comparison with the case of ρ linear, where we have necessary and sufficient conditions for a value $\delta_l^* \in (0, +\infty)$ to be the (unique) optimal solution (see [3]).

For any $\rho_0 \in \mathbb{R}_+$, let $\mathcal{C}v_{\rho_0}$ and $\mathcal{C}x_{\rho_0}$ respectively denote the class of concave and convex functions $\rho : [0, +\infty) \to [0, +\infty)$ such that

$$\rho'(0) \equiv \lim_{t \to 0^+} \rho'(t) = \rho_0.$$

We notice that, for any fixed ρ_0, the class of linear functions such that $\rho'(0) = \rho_0$ consists of the only element $\rho(t) = \rho_0 t$, that we have considered till now and which we continue to refer to. The proofs of our results (stated in Theorems 3.1 and 3.2 below) are based on the following fact:

Lemma 3.1. *For any fixed $\rho_0 \in \mathbb{R}_+$ and for any $\delta \geq 0$,*

- $\mathcal{R}'_\rho(\delta) > \mathcal{R}'_l(\delta)$ *for any $\rho \in \mathcal{C}v_{\rho_0}$;*
- $\mathcal{R}'_l(\delta) > \mathcal{R}'_\rho(\delta)$ *for any $\rho \in \mathcal{C}x_{\rho_0}$.*

Proof. Since ρ concave (resp. convex) implies that

$$-\int_0^\infty \overline{G}(t+\delta)\rho''(t)dt > 0 \text{ (resp. } < 0),$$

the thesis follows.

Remark 3.1. We point out that the inequalities in Lemma 3.1 are strict. In fact, a non-strict inequality would imply $\rho''(t) = 0$ for any $t \in (0, +\infty)$. This fact, on its turn, in view of the twice-differentiability of ρ, would imply ρ to be linear. This circumstance has the following consequence (see proof of Theorems 3.1 and 3.2): if $\mathcal{R}'_l(\delta) = 0$ and $\rho \in \mathcal{C}v_{\rho_0} \cup \mathcal{C}x_{\rho_0}$, δ cannot be an extremal point for $\mathcal{R}'_\rho$.

The following result is a straight consequence of Lemma 3.1:

Corollary 3.1. *If $\rho \in \mathcal{C}v_{\rho_0}$ and $\delta_\rho^* \in (0, +\infty)$ is the optimal burn-in time, it belongs to the set*

$$D = \{\delta \in (0, +\infty) \mid \mathcal{R}_l'(\delta) < 0\}.$$

Similarly, if $\rho \in \mathcal{C}x_{\rho_0}$ and $\delta_\rho^* \in (0, +\infty)$ is the optimal burn-in time, it belongs to the set

$$I = \{\delta \in (0, +\infty) \mid \mathcal{R}_l'(\delta) > 0\}.$$

This corollary is of straight application in the following Theorems 3.1 and 3.2, which provide regions for values of the possible optimal burn-in time in the concave and convex cases. Such regions are determined in terms of maxima and minima of $\mathcal{R}_l$. Results about the latter point have been provided in [3]. The proof is based on the previous results: we will use in particular Lemma 2.1, to qualitatively establish the shape of $\mathcal{R}_l'$, Theorem 2.1 and its Corollary 2.1, to state the uniqueness of δ_{min} and δ_l^* and to determine inflection points, affecting the set of the possible values of δ_ρ^*, and Lemma 3.1, to display the graph of $\mathcal{R}_\rho'$ over or under the one of $\mathcal{R}_l'$.

In Theorems 3.1 and 3.2 the case of $\overline{G}$ bathtub or upside down bathtub is considered, that is when r' changes sign only once. In the next section, we will see how to obtain this kind of results when such hypothesis is weakened.

Theorem 3.1. *Let $\rho \in \mathcal{C}v_{\rho_0}$.*

- *If $\mathcal{R}_l'(0) > 0$, then $\delta_\rho^* \in (\delta_l^*, \delta_{min}) \cup \{+\infty\}$;*
- *if $\mathcal{R}_l'(0) < 0$, then $\delta_\rho^* \in (0, \delta_{min}) \cup (\delta_l^*, +\infty]$.*

Proof. In view of Lemma 2.1, several situations may occur, depending on the shape of $\overline{G}$ and the sign of $\mathcal{R}_l'(0)$. We can distinguish four different cases:

1. $\overline{G}$ bathtub, $\mathcal{R}_l'(0) > 0$;
2. $\overline{G}$ bathtub, $\mathcal{R}_l'(0) < 0$;
3. $\overline{G}$ upside down bathtub, $\mathcal{R}_l'(0) > 0$;
4. $\overline{G}$ upside down bathtub, $\mathcal{R}_l'(0) < 0$.

1. We recall that $\overline{G}$ bathtub means that an only change point $\sigma \in [0, +\infty]$ exists such that $r'(\delta) \leq 0$ for any $\delta \in (0, \sigma)$ and $r'(\delta) \geq 0$ for any $\delta \in (\sigma, +\infty)$. By

Lemma 2.1, for any $\delta \in (0,\sigma)$, $\mathcal{R}_l''(\delta) \leq 0$; therefore $\mathcal{R}_l'(\delta)$ may eventually change sign in a point $\delta_l^* \in (0,\sigma)$.

When $\mathcal{R}_l'(\delta) < 0$, its monotonicity character is not determined, but $\mathcal{R}_l'$ cannot become positive, since, if, for some $\bar{\delta}$, $\mathcal{R}_l'(\bar{\delta}) = 0$, it would be $\mathcal{R}_l''(\bar{\delta}) \leq 0$ again. Thus, in $(0,\sigma)$, $\mathcal{R}_l'$ may change sign at most once and therefore $\mathcal{R}_l$ can have (by Theorem 2.1) at most one point of optimum.

If $\mathcal{R}_l'(\sigma) < 0$, by Lemma 2.1, $\mathcal{R}_l''(\sigma) > 0$. Again $\mathcal{R}_l'(\delta)$ may change its sign in a point $\delta_{min} \in (\sigma,+\infty)$. When $\mathcal{R}_l'(\delta) > 0$, for $\delta \in (\sigma,+\infty)$, the monotonicity character of $\mathcal{R}_l'$ is not determined, but, for such δ's, it cannot be $\mathcal{R}_l'(\delta) < 0$, since, as soon as, for some $\bar{\delta}$, $\mathcal{R}_l'(\delta) = 0$, it would be $\mathcal{R}_l''(\delta) \geq 0$ and therefore $\mathcal{R}_l'$ increasing again. Hence, for any $\delta > \delta_{min}$, $\mathcal{R}_l'(\delta) > 0$. The same happens if $\mathcal{R}_l'(\sigma) > 0$: for any $\delta > \sigma$, $\mathcal{R}_l'(\delta) > 0$.

Thus, in $(\sigma,+\infty)$, $\mathcal{R}_l$ can only have, again by Theorem 2.1, at most one point of optimum or alternatively one (rising) inflection point in σ.

By Lemma 3.1, if $\delta_\rho^* \in (0,+\infty)$, it has to be $\delta_\rho^* \in (\delta_l^*, \delta_{min})$; otherwise we set $\delta_\rho^* = +\infty$.

2. For the argument presented in the previous point, in $(0,\sigma)$ it must be $\mathcal{R}_l'(\delta) \leq 0$ and, because of Theorem 2.1, $\mathcal{R}_l$ has no points of inflection. In $(\sigma,+\infty)$, at most one point δ_{min} exists such that $\mathcal{R}_l'(\delta_{min}) = 0$.

   Therefore, by Lemma 3.1, if $\delta_\rho^* \in (0,+\infty)$, it must be $\delta_\rho^* \in (0,\delta_{min})$; otherwise, by definition, $\delta_\rho^* = +\infty$.

   We recall that in this case, by Proposition 2.1, it is $\delta_l^* = +\infty$, therefore the condition $\delta_\rho^* \in (0,\delta_{min}) \cup \{+\infty\}$ coincides with the thesis

$$\delta_\rho^* \in (0,\delta_{min}) \cup (\delta_l^*, +\infty].$$

3. In $(0,\sigma)$, by Lemma 2.1, it cannot be $\mathcal{R}_l'(\delta) < 0$. Therefore, by Theorem 2.1, $\mathcal{R}_l$ has neither extremal nor inflection points. In $(\sigma,+\infty)$, it exists at most one point δ_l^* such that $\mathcal{R}_l'(\delta_l^*) = 0$, that is a point of optimum for $\mathcal{R}_l$.

   For $\delta > \delta_l^*$, $\mathcal{R}_l'(\delta) < 0$. Therefore, $\delta_\rho^* \in (\delta_l^*, +\infty]$.

Since, in this case, $\delta_{min} = +\infty$, the previous condition coincides with the thesis $\delta^*_{cv} \in (\delta^*_l, \delta_{min}) \cup \{+\infty\}$.

4. In $(0,\sigma)$, $\mathcal{R}''_l(\delta) \geq 0$ till a possible δ_{min} such that $\mathcal{R}'_l(\delta_{min}) = 0$.

   For $\delta \in (\delta_{min}, \sigma)$, $\mathcal{R}'_l(\delta) > 0$.

   If $\mathcal{R}'_l(\sigma) < 0$, $\mathcal{R}'_l(\delta) < 0$ for any $\delta \in (\sigma, +\infty)$; in such a case $\delta^*_l = \delta_{min} = +\infty$.

   Again in $(\sigma, +\infty)$, if $\mathcal{R}'_l(\sigma) > 0$, $\mathcal{R}''_l(\delta) \leq 0$.

   At most one point $\delta^*_l \in (\sigma, +\infty)$ may exist such that $\mathcal{R}'_l(\delta^*_l) = 0$. For any $\delta > \delta^*_l$, $\mathcal{R}'_l(\delta) < 0$.

   Therefore, $\delta^*_\rho \in (0, \delta_{min}) \cup (\delta^*_l, +\infty]$.

Remark 3.2. In the previous proof, we neglect to consider the case when $\mathcal{R}'_l(\sigma) = 0$. Only when $\overline{G}$ is upside down bathtub and $\mathcal{R}'_l(0) < 0$ (see point 4.), $\mathcal{R}'_l(\sigma) = 0$ implies that σ is a falling inflection point for $\mathcal{R}_l$ and therefore it must be excluded from the set of the possible values for δ^*_ρ.

An analogous result holds when ρ is convex. The proof is analogous to the case when ρ is concave and therefore it is omitted.

Theorem 3.2. *Let* $\rho \in \mathcal{C}x_{\rho_0}$.

- *If* $\mathcal{R}'_l(0) > 0$, *then* $\delta^*_\rho \in (0, \delta^*_l) \cup (\delta_{min}, +\infty]$;
- *if* $\mathcal{R}'_l(0) < 0$, *then* $\delta^*_\rho \in (\delta_{min}, \delta^*_l) \cup \{+\infty\}$.

Remark 3.3. Theorem 3.2 presents some differences from Theorem 3.1. Such differences are due to the opposite point of view that we adopt in comparing the present case to the linear one. One may imagine that we observe a symmetric behaviour for the concave and convex cases. In fact the theses are in some way inverted with respect to Theorem 3.1, as it may be intuitive. Less intuitively, we notice that also the roles of δ_{min} and δ^*_l are inverted.

Remark 3.4. Also for $\rho \in \mathcal{C}x_{\rho_0}$, we neglect to consider in Theorem 3.2 the case when $\mathcal{R}'_l(\sigma) = 0$. Only when $\overline{G}$ is bathtub and $\mathcal{R}'_l(0) > 0$ (notice again that such conditions are inverted with respect to the previous case, when $\rho \in \mathcal{C}v_{\rho_0}$), $\mathcal{R}'_l(\sigma) = 0$ implies that σ is a rising inflection point for $\mathcal{R}_l$ and therefore it must be excluded from the set of the possible values for δ^*_ρ.

4. Multiple change points distributions

So far, we considered the case when r' changes sign at most once. We suitably extend Theorems 3.1 and 3.2 to the case when $\overline{G}$ has multiple change points $\sigma_1, \ldots, \sigma_n$.

Theorems 3.1 and 3.2 are based on the uniqueness of δ_l^*, in the case when r' has at most one change of sign. Theorem 2.1 instead concerns distributions that are not bathtub nor upside down bathtub, having multiple change points. In the case considered therein, $\mathcal{R}_l$ admits at most $n+1$ extremal points, $\delta_0, \ldots, \delta_n \in (0, +\infty)$, $\delta_0 < \cdots < \delta_n$, and eventual inflection points only at $\sigma_1, \ldots, \sigma_n$. More in particular, $\mathcal{R}_l$ admits at most one extremal point in any interval (σ_i, σ_{i+1}) and at most one optimal point in any interval $(\sigma_{i-1}, \sigma_{i+1})$ and therefore at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$ optimal points on $(0, +\infty)$.

By iteratively applying Theorems 3.1 and 3.2 to any interval $(\sigma_{i-1}, \sigma_{i+1})$, we obtain bounds for δ_ρ^* also if G is not bathtub nor upside down bathtub.
To perform such an iteration, we have to extract from $\delta_0, \ldots, \delta_n$ two sequences: the one of the local optima, $\{\delta_k^*\}$, and the one of the local minima, $\{\delta_k^{min}\}$.

If $\mathcal{R}_l'(0) > 0$, then

$$\delta_0 \equiv \min(\{\delta \mid \mathcal{R}_l'(\delta) = 0\} \setminus \{\sigma_1, \ldots, \sigma_n\})$$

is a maximum point, and we set

$$\delta_1^* = \delta_0;$$

$$\delta_1 \equiv \min(\{\delta \mid \mathcal{R}_l'(\delta) = 0\} \setminus \{\delta_0, \sigma_1, \ldots, \sigma_n\})$$

is a minimum point, and we set

$$\delta_1^{min} = \delta_1;$$

and so on.
$\mathcal{R}_l$ admits at most $n+1$ extremal points, but not necessarily $n+1$, meaning that it may happen that for a certain index $\bar{n} \le n$, we find that

$$\delta_{\bar{n}} \equiv \min(\{\delta \mid \mathcal{R}_l'(\delta) = 0\} \setminus \{\delta_0, \ldots, \delta_{\bar{n}-1}, \delta_0, \sigma_1, \ldots, \sigma_n\}) = +\infty.$$

If $\bar{n}$ is even, we set

$$\delta_k^* = +\infty, \quad \delta_k^{min} = +\infty, \quad \text{for } k = \frac{\bar{n}}{2} + 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1;$$

if $\bar{n}$ is odd, it will be

$$\delta_k^{min} = +\infty, \quad \text{for } k = \frac{\bar{n}+1}{2}, \dots, \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad \delta_{k+1}^* = +\infty, \quad \text{for } k = \frac{\bar{n}+1}{2}, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

Symmetrically, if $\mathcal{R}_l'(0) < 0$, the above-defined δ_0 is a minimum point, and we set

$$\delta_1^{min} = \delta_0;$$

δ_1 is instead a maximum point and we set

$$\delta_1^* = \delta_1,$$

and so on.

If $\mathcal{R}_l$ admits less than $n+1$ extremal points, that is if $\bar{n} \le n$ exists, such that $\delta_{\bar{n}} = +\infty$, when $\bar{n}$ is even, we set

$$\delta_k^* = +\infty, \quad \delta_k^{min} = +\infty, \quad \text{for } k = \frac{\bar{n}}{2} + 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor + 1;$$

when $\bar{n}$ is odd, it will be

$$\delta_k^* = +\infty, \quad \text{for } k = \frac{\bar{n}+1}{2}, \dots, \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad \delta_{k+1}^{min} = +\infty, \quad \text{for } k = \frac{\bar{n}+1}{2}, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

We are now in a position to state the following theorem.

Theorem 4.1. *Let $\rho \in \mathcal{C}v_{\rho_0}$.*

If $\mathcal{R}_l'(0) > 0$, then

$$\delta_\rho^* \in \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} (\delta_i^*, \delta_i^{min}) \cup \{+\infty\};$$

if $\mathcal{R}_l'(0) < 0$, then

$$\delta_\rho^* \in (0, \delta_1^{min}) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (\delta_i^*, \delta_{i+1}^{min}) \right) \cup (\delta^*_{\lfloor \frac{n}{2} \rfloor + 1}, +\infty].$$

Let $\rho \in \mathcal{C}x_{\rho_0}$.

If $\mathcal{R}_l'(0) > 0$, then

$$\delta_\rho^* \in (0, \delta_1^*) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (\delta_i^{min}, \delta_{i+1}^*) \right) \cup (\delta^{min}_{\lfloor \frac{n}{2} \rfloor + 1}, +\infty];$$

if $\mathcal{R}_l'(0) < 0$, then

$$\delta_\rho^* \in \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor + 1} (\delta_i^{min}, \delta_i^*) \cup \{+\infty\}.$$

5. Reward functions with mission time

Till now, we have considered the special case where the reward function is the one in Eq. (0.1). We want to point out that however our results are extendible, at the cost of slight changes, to other cases. In this section, we describe in particular a model with the presence of a mission time τ, whose reward function turns out to be

$$R_{\tau,\rho}(t,\delta) = \rho(t-\delta)\mathbf{1}_{\{t>\tau+\delta\}} + \tilde{k}\mathbf{1}_{\{\delta<t<\tau+\delta\}} + k\mathbf{1}_{\{t<\delta\}} + c\min(\delta,t). \tag{5.1}$$

First, along the line of [3, Proposition 2], we give conditions for finding the locally optimal burn-in times for the version of the reward function $R_{\tau,\rho}(t,\delta)$ in the case when ρ is linear (see Proposition 5.1 below), that we denote by $R_{\tau,l}(t,\delta)$.
In doing that the function $\dfrac{g(\tau+\delta)}{g(\delta)}$ plays an additional role to the one of r in determining the monotonicity character of $\mathcal{R}'_{\tau,l}$ (see Remark 5.1 below), therefore we lose the monotonicity relationship between r and of $\mathcal{R}'_l$ we have had till now (see Lemma 2.1). As a consequence, even for linear ρ, we do no manage in determining a maximum number of extremal points for $\mathcal{R}'_{\tau,\rho}$.

In particular, if we come back again to consider $\overline{G}$ bathtub or upside down bathtub, we lose the uniqueness of δ_l^*. Therefore, for the discussion of the present case, we draw inspiration from Theorem 4.1, providing for the eventuality of more than one optimal point. However the result of this section differs from it for what concerns the other hypotheses and assumptions on the number of extremal and inflection points and their localization with respect to the change points. In order to prove the following Theorem 5.1, we can rely only on a result analogous to Lemma 3.1, holding for $\mathcal{R}'_{\tau,l}$ and $\mathcal{R}'_{\tau,\rho}$ as well.

Even if some hypotheses satisfied by the previous reward function are lost, still we can provide bounds for the solutions of the convex or concave case based on the solutions in the linear case. However, in applying the theorem, we have to pay attention to the fact that, even if part of the thesis is still satisfied, we lose any link with ageing properties represented by the bathtub or UB shape of $\overline{G}$ or by the multiple change points form of r. In fact, the reward function in Eq. (5.1) does not allow us to find immediate relations with them. Therefore the proof of the following Theorem 5.1 only can use an analogous relation of Lemma 3.1.

Correspondingly to Eq. (5.1), we have

$$\mathcal{R}_{\tau,\rho}(\delta) \equiv \mathbb{E}[R_{\tau,\rho}(T,\delta)] =$$

$$\tilde{k}[G(\tau+\delta) - G(\delta)] + kG(\delta) + \int_0^{+\infty} \rho(t)g(t+\delta)dt + c\delta\overline{G}(\delta) + c\int_0^{\delta} tg(t)dt \tag{5.2}$$

and

$$\mathcal{R}'_{\tau,\rho}(\delta) = (k-\tilde{k})g(\delta) - \int_0^{+\infty} \rho''(t)\overline{G}(t+\delta)dt + (c-\rho_0)\overline{G}(\delta) + \tilde{k}g(\tau+\delta). \tag{5.3}$$

Remark 5.1.

- $r(\delta)$ and $\dfrac{g(\tau+\delta)}{g(\delta)}$ decreasing in δ and $\mathcal{R}'_{\tau,\rho}(\delta) < 0$ imply $\mathcal{R}''_{\tau,\rho}(\delta) > 0$.
- $r(\delta)$ and $\dfrac{g(\tau+\delta)}{g(\delta)}$ increasing in δ and $\alpha(\delta)\mathcal{R}'_{\tau,\rho}(\delta) > 0$ imply $\mathcal{R}''_{\tau,\rho}(\delta) < 0$.

For this reward function, even in the linear case, we have not warranted the uniqueness of the optimum, as the following proposition let argue.

Proposition 5.1. *Let ρ be linear. $\delta^* > 0$ is a locally optimal burn-in time if and only if*

$$r(\delta^*) = \frac{\rho_0 - c}{k - \tilde{k}\left(1 - \dfrac{g(\tau+\delta^*)}{g(\delta^*)}\right)} \tag{5.4}$$

and

$$\begin{cases} \dfrac{g(\tau+\delta^*)}{g(\delta^*)} < \left(1 - \dfrac{k}{\tilde{k}}\dfrac{r(\delta^*)-\alpha(\delta^*)}{r(\delta^*)-\alpha(\tau+\delta^*)}\right) & \text{if } r(\delta^*) < \alpha(\tau+\delta^*) \\ \\ \dfrac{g(\tau+\delta^*)}{g(\delta^*)} > \left(1 - \dfrac{k}{\tilde{k}}\dfrac{r(\delta^*)-\alpha(\delta^*)}{r(\delta^*)-\alpha(\tau+\delta^*)}\right) & \text{if } r(\delta^*) > \alpha(\tau+\delta^*) \end{cases} \tag{5.5}$$

Proof. Maximizing the expected reward $\mathcal{R}_{\tau,l}(\delta)$ with respect to δ is equivalent to finding $\delta > 0$ such that

$$\begin{cases} \mathcal{R}'_{\tau,l}(\delta) = 0 \\ \mathcal{R}''_{\tau,l}(\delta) < 0 \end{cases}$$

In the linear case

$$\mathcal{R}'_{\tau,l}(\delta) = (k-\tilde{k})g(\delta) + (c-\rho_0)\overline{G}(\delta) + \tilde{k}g(\tau+\delta),$$

that straightly leads us to Eq. (5.4).

$$\mathcal{R}''_{\tau,l}(\delta) = (k-\tilde{k})g'(\delta) + (\rho_0 - c)g(\delta) + \tilde{k}g'(\tau+\delta).$$

Therefore $\mathcal{R}''_{\tau,l}(\delta) < 0$ if and only if

$$(\tilde{k} - k)\alpha(\delta) + \rho'(0) - c - \tilde{k}\alpha(\tau + \delta)\frac{g(\tau + \delta)}{g(\delta)} < 0.$$

Since we are interested in computing $\mathcal{R}''_{\tau,l}(\delta)$ for those δ's already satisfying Eq. (5.4), we obtain

$$(\tilde{k} - k)(\alpha(\delta) - r(\delta)) + \tilde{k}(r(\delta) - \alpha(\tau + \delta))\frac{g(\tau + \delta)}{g(\delta)} < 0.$$

By discussing the sign of $r(\delta) - \alpha(\tau + \delta)$, we get the condition (5.5).

Remark 5.2. Since $\dfrac{g(\tau + \delta)}{g(\delta)} \geq 0$ for any $\tau, \delta \geq 0$, the optimization problem has no solutions if $\alpha(\delta) < r(\delta) < \alpha(\tau + \delta)$, while condition (5.5) is always satisfied if $\alpha(\tau + \delta) < r(\delta) < \alpha(\delta)$.

Also for the expected reward in Eq. (5.2), Lemma 3.1 and Corollary 3.1 hold. More precisely:

1. for any fixed $\rho_0 \in \mathbb{R}$ and for any $\delta \geq 0$,
   - $\mathcal{R}'_{\tau,\rho}(\delta) > \mathcal{R}'_{\tau,l}(\delta)$ for any $\rho \in \mathcal{C}v_{\rho_0}$;
   - $\mathcal{R}'_{\tau,l}(\delta) > \mathcal{R}'_{\tau,\rho}(\delta)$ for any $\rho \in \mathcal{C}x_{\rho_0}$;
2.
   - if $\rho \in \mathcal{C}v_{\rho_0}$ and $\delta^*_\rho \in (0, +\infty)$ is the optimal burn-in time, it belongs to the set
   $$D = \{\delta \in (0, +\infty) \mid \mathcal{R}'_{\tau,l}(\delta) < 0\};$$
   - if $\rho \in \mathcal{C}x_{\rho_0}$ and $\delta^*_\rho \in (0, +\infty)$ is the optimal burn-in time, it belongs to the set
   $$I = \{\delta \in (0, +\infty) \mid \mathcal{R}'_{\tau,l}(\delta) > 0\}.$$

Suitably modified, an analog of Theorem 4.1 holds even under weaker hypotheses. We lose the statement about the non-existence of the inflexion points and δ_k's just are the points where $\mathcal{R}'_{\tau,\rho}(\delta) = 0$, without relation with the change points of r, σ_k's.

We also notice that, again, also for the reward function in Eq. (5.1), in view of finding bounds for the possible solutions of concave or convex burn-in problems, we need not to find the global optimum of the linear problem, but only the local ones.

So like Theorem 4.1, the following theorem uses local maximum and minimum points for $\mathcal{R}_{\tau,l}$ to determine the set where local maximum points of $\mathcal{R}_{\tau,\rho}$ can fall.

We denote by $h \in \mathbb{N}$ the number of solutions of Eq.'s (5.4) and (5.5). Now, the sequences $\{\delta_i^*\}$, $\{\delta_i^{min}\}$ respectively consist of the solutions of Eq.'s (5.4) and (5.5) and of the ones of Eq. (5.4) satisfying Eq. (5.5) with the reverted inequality signs.

Theorem 5.1. *Let* $\rho \in \mathcal{C}v_{\rho_0}$*;*

if $\mathcal{R}'_{\tau,l}(0) > 0$*, then*

$$\delta_\rho^* \in \bigcup_{i=1}^{h} (\delta_i^*, \delta_i^{min}) \cup \{+\infty\} \setminus \mathcal{D}_{fl}^d; \tag{5.6}$$

if $\mathcal{R}'_l(0) < 0$*, then*

$$\delta_\rho^* \in (0, \delta_1^{min}) \cup \left(\bigcup_{i=1}^{h-1} (\delta_i^*, \delta_{i+1}^{min}) \right) \cup (\delta_h^*, +\infty] \setminus \mathcal{D}_{fl}^d.$$

Let $\rho \in \mathcal{C}x_{\rho_0}$*;*

if $\mathcal{R}'_{\tau,l}(0) > 0$*, then*

$$\delta_\rho^* \in (0, \delta_1^*) \cup \left(\bigcup_{i=1}^{h-1} (\delta_i^{min}, \delta_{i+1}^*) \right) \cup (\delta_h^{min}, +\infty] \setminus \mathcal{D}_{fl}^a; \tag{5.7}$$

if $\mathcal{R}'_{\tau,l}(0) < 0$*, then*

$$\delta_\rho^* \in \bigcup_{i=1}^{h} (\delta_i^{min}, \delta_i^*) \cup \{+\infty\} \setminus \mathcal{D}_{fl}^a.$$

Notice that, in Eq.'s (5.6), (5.7), it can be $\delta_h^{min} \in (\delta_h^*, +\infty)$ or $\delta_h^{min} = +\infty$, while, in the other two equations, it has necessarily to be $\delta_h^{min} \in (\delta_{h-1}^*, \delta_h^*)$.

6. Conclusions

In [3], some results are provided, linking ageing to the solution of the optimal burn-in time problem for the cost structure described by Eq. (0.1). We chose therein such a reward function, presenting a form that allows us to highlight connections between the optimal burn-in problem and relevant ageing properties.

We start from such results, that are, under some aspects, preliminary for the analytical study we carried on in the present paper. We provide here results about the optimal burn-in time under a given cost structure, with ρ linear and a hazard rate with a finite number of monotonicity change points. When ρ is linear, at most one locally optimal

burn-in time falls in any time interval where r is monotonic. As a particular instance of this result, under the hypothesis that $\overline{G}$ is bathtub or upside down bathtub, we get the uniqueness of the optimal burn-in time.
Determining the local maximum points under the hypothesis of ρ's linearity serves as a basis for establishing bounds for the set where the local maximum points may fall, in the cases when ρ is concave or convex. The locally optimal times when ρ is linear can be determined by applying the condition provided in [3, Proposition 2].

In the bathtub or upside down bathtub case, we can also state a more schematic criterion to establish the existence or the position of δ_l^*, provided that we know the sign of $\mathcal{R}_l'(\sigma)$ and $r'(\varepsilon)$ (with $0 < \varepsilon < \sigma$):

- $\mathcal{R}_l'(\sigma) r'(\varepsilon) \leq 0$ implies that both δ_l^* and δ_{min} do not exist in $(0, +\infty)$;
- $\mathcal{R}_l'(\sigma) < 0,\ r'(\varepsilon) < 0$ imply $\delta_l^* \in (0, \sigma)$;
- $\mathcal{R}_l'(\sigma) > 0,\ r'(\varepsilon) > 0$ imply $\delta_{min} \in (0, \sigma)$.

By combining it with Theorem 3.1 or 3.2, we obtain conditions for existence of δ_ρ^* also in the case when ρ is concave or convex.

Such a criterion can be straightly extended to the case of r with multiple change points, to establish the existence or the position of a local optimum point δ_i^* in any interval $(\sigma_{i-1}, \sigma_{i+1})$, $i = 1, \ldots, n$, provided that we know the sign of $\mathcal{R}_l'(\sigma_i)$ and $r'(\sigma_i - \varepsilon)$ (with $0 < \varepsilon < \sigma_i - \sigma_{i-1}$).

We also show how our results, obtained under a particular cost structure, can be used as a starting point for obtaining analogous ones under a modification of the cost structure, where a surcharge, associated with a mission time, is inserted.

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