A Generalized Stochastic Knapsack Problem with Application in Call Admission Control

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1 Introduction

In the classical *knapsack problem*, a knapsack of capacity C is given, together with K classes of objects. For every $k = 1, \ldots, K$, each object of the class k has a size b_k and an associated reward r_k. The objects can be placed into the knapsack as long as the sum of their sizes does not exceed the capacity C. The problem consists in placing the objects inside the knapsack so as to maximize the total reward.

Among the various extensions to a stochastic framework available in the literature (see, e.g., [7,5,3]), we consider the *stochastic knapsack problem* proposed in [7]. In such a model, objects belonging to each class become available randomly, according to exponentially-distributed inter-arrivals times with means depending on the class and on the state of the knapsack. Each object has a sojourn time independent from the sojourn times of the other objects and described by a class-dependent distribution. If put into the knapsack, an object from class k generates revenue at a positive rate r_k. Let $n_k \geq 0$ denote the number of objects of class k that are currently inside the knapsack. Then one has the linear constraint

$$\sum_{k \in K} n_k b_k \leq C \, . \tag{1}$$

The problem consists in finding a policy that maximizes the average revenue, by accepting or rejecting the arriving objects in dependence of the current state of the knapsack.

The stochastic knapsack problem that we have just described has application, e.g., in *Call Admission Control* (*CAC*) for telecommunication networks. In such a context, the objects are requests of connections coming from K different classes of

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users, each with a bandwidth requirement $b_k,\ k = 1, \dots, K$, and a distribution for its duration. In CAC problems, often the constraint (1) arises as a linearization of the nonlinear constraint

$$\sum_{k \in K} \beta_k(n_k) \leq C\,, \tag{2}$$

where the $\beta_k(\cdot)$ are nonlinear nonnegative functions. The model in which the linear constraint (1) is replaced by the nonlinear one (2) is known in the literature as the *generalized stochastic knapsack problem* [1] [4]. In call admission control, the coefficients b_k of the linearized constraint are called *effective bandwidths* [6, Chapter 1].

The sets

$$\Omega_{FR} := \{(n_1, \dots, n_K) \in \mathbb{N}_0^K : \sum_{k \in K} n_k b_k \leq C\}, \tag{3}$$

in the linear case, and

$$\Omega_{FR} := \{(n_1, \dots, n_K) \in \mathbb{N}_0^K : \sum_{k \in K} \beta_k(n_k) \leq C\} \tag{4}$$

in the nonlinear case, are called *feasibility regions*. In the context of admission control they model subsets of the call space $\{(n_1, \dots, n_K) \in \mathbb{N}_0^K\}$, where given *Quality of Service* (*QoS*) constraints are satisfied.

In general, finding optimal policies is a difficult combinatorial optimization task both for the stochastic knapsack problem [6, Chapter 4] and for the generalized stochastic one [1,2]. The a-priori knowledge of structural properties of the (unknown) optimal policies is useful to find the solutions or, at least, good suboptimal policies. For two classes of objects and the linear constraint (1), structural properties were derived in [7] for the optimal policies belonging to the family of *coordinate-convex policies*. Such properties restrict the K-tuple $(n_1, \dots, n_K)$ to suitable subsets of the feasibility region Ω_{FR}. Some extensions to nonlinearly-constrained feasibility regions of the structural results obtained in [7] for linearly-constrained ones were derived in [2] and other structural results were obtained in [1].

2 Problem formulation

Let $\mathbf{n}$ denote the vector $(n_1, \dots, n_k)$. For each class $k = 1, \dots, K$, the inter-arrival time is exponentially distributed with mean value $1/\lambda_k(n_k)$. The sojourn times of the accepted objects are independent and identically distributed (i.i.d.) with mean values $1/\mu_k,\ k = 1, \dots, K$. At the time of its arrival, each object is either accepted or rejected, according to a coordinate-convex *policy*, defined as follows [6, p. 116].

[1] Note that this is different from the “generalized stochastic knapsack” considered in [6, Chapter 3].

Definition 2.1 *A nonempty set $\Omega \subseteq \Omega_{FR} \subset \mathbb{N}_0^K$ is coordinate-convex iff it has the following property: for every $\mathbf{n} \in \Omega$ with $n_k > 0$ one has $\mathbf{n} - \mathbf{e}_k \in \Omega$, where $\mathbf{e}_k$ is a K-dimensional vector whose k-th component is 1 and the other ones are 0. The coordinate-convex policy associated with a coordinate-convex set Ω admits an arriving object iff after its insertion one has $\mathbf{n} \in \Omega$.*

Note that by (3) or (4), the set Ω_{FR} is itself coordinate-convex. As there is a one-to-one correspondence between coordinate-convex sets and coordinate-convex policies, in the following we use the symbol Ω to denote either a coordinate-convex set or a coordinate-convex policy.

The objective to be maximized in the set $\mathcal{P}(\Omega_{FR})$ of coordinate-convex subsets of Ω_{FR} is given by

$$J(\Omega) = \sum_{\mathbf{n} \in \Omega} (\mathbf{n} \cdot \mathbf{r}) P_\Omega(\mathbf{n}) \,, \tag{5}$$

where $\mathbf{r}$ denotes the vector $(r_1, \ldots, r_K)$ and $P_\Omega(\mathbf{n})$ is the steady-state probability that the current content of the knapsack is $\mathbf{n}$. As Ω is coordinate-convex, one can show that $P_\Omega(\mathbf{n})$ takes on the product-form expression

$$P_\Omega(\mathbf{n}) = \frac{\prod_{i=1}^K q_i(n_i)}{\sum_{\mathbf{n} \in \Omega} \prod_{i=1}^K q_i(n_i)} \,, \text{ where } q_i(n_i) := \frac{\prod_{j=0}^{n_i - 1} \lambda_i(j)}{n_i! \mu_i^{n_i}} \,. \tag{6}$$

Due to (6), in general the objective (5) is nonlinear. What makes the problem difficult is that, given any two coordinate-convex sets $\Omega_1, \Omega_2 \subseteq \Omega_{FR}$, in general the relationship $\Omega_1 \subseteq \Omega_2$ does not imply $J(\Omega_1) \leq J(\Omega_2)$.

3 Contributions

For generalized stochastic knapsack problems with two classes of objects (which model CAC with two classes of users), in [1] we derived for the optimal coordinate-convex policies structural properties that do not depend on the revenue ratio $R := r_2/r_1$. In [2], instead, we obtained properties that do depend on it. In the present work, we develop the investigation in the following directions.

(i) We analyze the optimal choices for some parameters used by a criterion proposed in [1] to improve certain suboptimal coordinate-convex policies. The criterion is based on the removal or addition of rectangular subregions near suitably-defined corner points.

(ii) We propose a greedy algorithm of approximate solution, based on the optimal choice of the parameters in (i). The related simulations show an improvement of the objective over the numerical results derived in [1].

(iii) We address some relationships between general structural properties of the optimal coordinate-convex policies and the greedy algorithm in (ii). In particular, we extend [1, Theorem III.6] by proving that the coordinate-convex set associated with an optimal coordinate-convex policy has a nonempty intersection with the upper boundary $(\partial \Omega_{FR})^+$ of the feasibility region Ω_{FR} (see

Fig. 1), independently of the number of its corner points.

(iv) We exploit another general structural property of the optimal coordinate-convex policies to initialize the greedy algorithm in (ii).

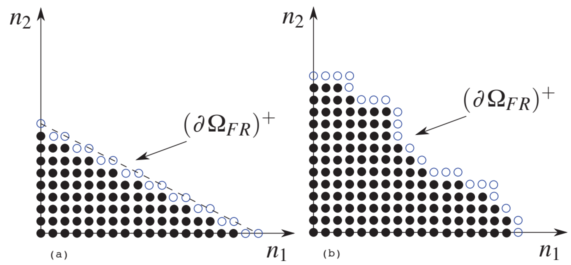

Fig. 1. The upper boundary $(\partial\Omega_{FR})^+$ of a feasibility region Ω_{FR} with two classes of objects in the case of (a) a linearly-constrained Ω_{FR} (stochastic knapsack) and (b) a nonlinearly-constrained Ω_{FR} (generalized stochastic knapsack).

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