Demographic risk transfer: is it worth for annuity providers?

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Abstract

Longevity risk transfer is a popular choice for annuity providers such as pension funds. This paper formalizes the trade-off between the cost and the risk relief of such choice, when the annuity provider uses value-at-risk to assess risk. Using first-order approximations we show that, if the transfer is fairly priced and the aim of the fund is to maximize returns, the funds' alternatives can be represented in the plane expected return-VaR. We build a risk-return frontier, along which the optimal transfer choices of the fund are located and calibrated it to the 2010 UK annuity and bond market.

The activity of pension funds and their asset managers is likely to be stimulated by both the current and forecasted reduction in public provision of pensions. The high level of public deficit with respect to GDP in a number of developed countries offers to private providers of annuities a unique opportunity to develop their activity and increase their business. This opportunity goes hands in hands with major risks, linked to the overall economic and demographic situation and to the relative inadequacy of theoretical risk management tools to cope with the many facets of actual businesses. This endangers the growth and success perspectives of private annuity providers.

Longevity risk - i.e. the risk of unexpected improvements in survivorship - is by now perceived as an important threat to the safety of annuity providers, such as pension funds. These institutions run the risk of seeing their liabilities increase over time, because their members survive more than predicted, i.e. the actual survival rate is greater than the forecasted one. As of 2007, the exposure of pension funds and other annuity providers to unexpected improvements in life expectancy has been computed to be 400 billion USD for the US and UK, more than 20 trillion USD worldwide (see Loeys, Panigirtzoglou and Ribeiro (2007)).

Annuity providers are also exposed to financial risks on both assets and liabilities, as soon as the latter are fairly evaluated. In principle, accounting for

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financial and longevity risk, while looking at both assets and liabilities, seems to be a very hard task. Even once this measuring task is addressed, by using for instance a fairly simple model for longevity appraisal, coupled with an equally simple model for financial risk appraisal, able to provide closed form expressions for risk measurement, other problems remain. They concern management of those risks.

An important management problem, which we explore in this paper, is whether an annuity provider should better transfer longevity risk to a reinsurer or a special purpose vehicle - as most of the recent deals do - or remaining exposed to it, while saving on the costs of the transfer. Even if the trade-off is obvious, assessing its impact without resorting to extensive use of simulations is not easy. This is due to the aforementioned difficulties in measuring longevity risk on the one side, and to the subtle interactions between longevity and financial risk on the other. These interactions arise if one considers liability management only, since the value of the liabilities is subject to interest rate risk, and - a fortiori - if one considers asset and liability management, as we do.

To anticipate on our results, this paper formalizes the trade-off between the cost and the risk relief of such choice, when the annuity provider uses value-at-risk to assess risk. Using first-order approximations we show that, if the transfer is fairly priced and the aim of the fund is to maximize returns, the funds' alternatives can be represented in the plane expected return-VaR. We build a risk-return frontier, along which the optimal transfer choices of the fund are located. We disentangle the demographic and financial component of the overall funds' risk.

The paper is structured as follows: Section 1 formalizes our set up for both demographic and financial risk. Section 2 measures the effects of transferring versus retaining mortality risk, as well as the effects of financial risk. In order to do that, it considers also how demographic risk transfer can be fairly priced. Section 3 presents the conditions or asset strategies for maximizing returns. Section 4 spells out the consequent trade-offs between risk and return, after having introduced a VaR measure for the overall risk of assets and liabilities. Section 5 compares the results of expected return maximizing strategies. This amounts to computing the risk/return trade-off of the fund strategies and to disentangle the contributions of demographic and financial losses. Section 6 provides an example using financial and demographic data from the UK market. Section 7 concludes and compares with related research.

1 Set up

We consider the stylized case of a pension fund which has issued a single annuity on a head aged x. The fund can either

- transfer demographic risk to a reinsurer - which in turn hedges and prices it fairly - or
- suffer it without hedging

At the same time, the fund is supposed to maximize expected financial returns.

We assume that hedging of demographic risk on the part of the reinsurer and measurement of financial risk on the part of the fund is done up to first order discrepancies between actual and forecasted interest and survival rates. We perform a Delta analysis, but provide also the Gamma sensitivities, since the extension to second order hedges, or Delta-Gamma hedging, is quite obvious.

The issues to be addressed in order to formalize the fund strategies, how they affect the insurance and financial contracts bought or sold, and the trade-offs between costs and benefits they entail, are:

1. how to measure demographic and financial risk
2. the effects and cost of transferring demographic risk
3. the effects of choosing an optimal asset mix.

Once risk measuring is formalized, a number of possible fund strategies will emerge from the combination of transferring/non transferring demographic risk and managing the asset mix. We assume below that the fund performs the optimal asset allocation in order to maximize expected returns. The choice of such an objective function is done for illustrative purposes only. It affects only the number of bonds in the fund assets. Nowhere we take a view on whether the fund must maximize expected returns or follow another financial strategy. So, the choice criterion for the asset mix could be changed without modifying our method (but modifying the numerical examples).

In order to make measurement and management of demographic and financial risk feasible in closed form, we place ourselves in a standard, continuous-time framework. Consider a time interval $\mathcal{T} = [0, T], T < \infty$, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a multidimensional standard Wiener $W(\omega, t), t \in \mathcal{T}$. The space is endowed with the filtration generated by w, $\mathbf{F}^w = \{\mathcal{F}_t\}$. We adopt a stochastic extension of the classical Gompertz law for mortality description and we stick to the Hull-White model for interest-rate risk[1]. In doing so, we assume that the assets of the pension fund can be invested in bonds or kept as cash.

1.1 Demographic risk

Mortality and longevity risk are described by assuming that death occurs with an intensity which, instead of being deterministic as in the classical actuarial framework, is stochastic. This stochastic intensity - or stochastic-mortality - approach, which is by now quite well known in the literature, has the advantage of making description of age, period and cohort effects in mortality possible. If the intensity is described by processes simple enough (linear affine), the survival

[1] This is the framework we adopted for measuring and hedging - but not for transferring - the Delta-Gamma risk of the reserves/liabilities in Luciano, Regis, Vigna (2012a,b) . In those papers risk is either securitized through derivatives or internally offset through the natural hedge between life and death contracts.

function is known in closed form and can be calibrated using a parsimonious number of parameters. In order to reap these advantages, while keeping the distinction between age, period and cohort effects, we assume that the mortality intensity of a head aged x at calendar time t - which belongs to generation i - can be described by a so-called Ornstein-Uhlenbeck process, without mean reversion (OU):

$$d\lambda_i(t) = a_i\lambda_i(t)dt + \sigma_i dW_i(t),$$

where $a_i > 0$, $\sigma_i > 0$, W_i is a standard one-dimensional Brownian motion in W. In the notation we omit the dependence on x, since once calendar time and generation or gender are specified, age is uniquely determined. If the generation is captured by the year of birth, age x is linked to time and generation by the equality $x = t - i$.

This intensity extends - with the inclusion of a diffusive term - the classical Gompertz law

$$d\lambda_i(t) = a_i\lambda_i(t)dt,$$

where $a_i > 0$ is the rate of growth of the force of mortality.
Expected intensity increases over age:

$$\mathbb{E}_t(\lambda_i(t+\Delta t)) = \lambda_i(t)\exp(a_i\Delta t) = f_i(t, t+\Delta t) + \frac{\sigma_i^2}{2a_i^2}\left[1 - \exp\left(a_i\Delta t\right)\right]^2, \quad (1)$$

where $f_i(t, t+\Delta t)$ is the forward mortality intensity, i.e. the mortality intensity at time $t + \Delta t$, as forecasted at time t.

The instantaneous volatility is constant, while the overall variance increases exponentially in time:

$$\mathbb{V}ar_t(\lambda_i(t+\Delta t)) = -\frac{\sigma_i^2}{2a_i}\left[1 - \exp\left(2a_i\Delta t\right)\right]. \quad (2)$$

In order to capture longevity improvements over generations we assume that there is an intensity process for each generation. This dependence on the generation - which shows up through the index i - will enter through the parameter calibration. We calibrate one drift and one diffusion for each generation (and gender, obviously). This - together with the OU choice - makes the overall mortality model quite parsimonious[2].

On top of being parsimonious, the model provides a closed form for the survival probability of head x of generation i at any point in time t and up to any horizon T :

$$\begin{aligned} S_i(t,T) &= \mathbb{E}_t\left[\exp\left(-\int_t^T \lambda_i(s)ds\right)\right] = \\ &= \frac{S_i(0,T)}{S_i(0,t)}\exp\left[-X_i(t,T)I_i(t) - Y_i(t,T)\right], \end{aligned}$$

[2]If we model all the generations at the same time, in order to capture their correlation intensities, the number of parameters grows, but can still be kept under control: see Jevtic, Luciano, Vigna(2012).

where $S_i(0,T)$ and $S_i(0,t)$ are the survival frequencies at time 0

$$X_i(t,T) := \frac{\exp(a_i(T-t)) - 1}{a_i},$$

$$Y_i(t,T) := -\frac{\sigma_i^2 \left[1 - \exp\left(2a_i t\right)\right] X_i(t,T)^2}{4a_i},$$

$$I_i(t) := \lambda_i(t) - f_i(0,t).$$

and $I_i(t)$ - the difference between the actual mortality intensity of generation i at t and its forward value or forecast at time 0, $f_i(0,t)$ - is what we interpret as the *mortality or demographic risk factor*. It is the discrepancy between realization and forecast which makes the pension fund exposed to mortality risk.

Please notice that, since we are describing mortality and longevity risk, only the survival probabilities at the current date ($t = 0$) are known, while the probabilities which will be assigned at any future point in time ($t > 0$) are random variables. We will see below that this makes the reserves of the pension fund at any future point in time stochastic, and generates the demographic risk he has to cover. Randomness enters through the demographic factor $I_i(t)$ and affects the whole survival curve, namely $S_i(t,T)$ for every T.

1.2 Financial risk

In order to seize the effects of interest rate changes on assets and liabilities we need to select a model for financial risk. The natural choice is to assume that interest rates follow an Hull-and-White one-factor model. This is a standard choice in Financial modelling, able to provide us with closed form formulas for pricing and hedging, parsimonious but flexible enough to be popular in applications. In our context, it has the advantage of modelling risk in a fashion symmetric to demographic risk with an OU intensity. Indeed, the instantaneous interest rate in the Hull-White model has the following dynamics under a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$:

$$dr(t) = g(\theta - r(t))dt + \Sigma dW_F(t), \tag{3}$$

where $\theta, g > 0, \Sigma > 0$ and W_F is a univariate Brownian motion independent[3] of W_i for all i. θ is the long-run mean of the short-rate process, while the parameter g is the speed at which the current level of r is reverting to this value. As a consequence, the instantaneous rate has expectation and variance equal to

$$\mathbb{E}_t\left[r(t+\Delta t)\right] = r(t)e^{-g\Delta t} + \theta\left[1 - e^{-g\Delta t}\right], \tag{4}$$

$$\mathbb{V}ar_t(r(t+\Delta t)) = \frac{\Sigma^2}{2g}\left[1 - \exp\left(-2g\Delta t\right)\right]. \tag{5}$$

[3] Under the original measure we have $W = (W_1, W_2, ...W_N, W_F')$ where N is the maximum number of generations alive in $\mathcal{T}$, while W_F' is the Brownian motion which corresponds to W_F according to Girsanov' theorem. No arbitrage and completeness is assumed to hold in the financial market. Independency of financial and actuarial risk is preserved because of the diffusive nature of uncertainty: see Luciano, Regis, Vigna (2012a).

The corresponding zero-coupon bond price - if the bond is evaluated at t and has maturity T - is[4]

$$B(t,T) = \mathbb{E}_t^{\mathbb{Q}} \left[\exp \left(- \int_t^T r(s)ds \right) \right] =$$

$$= \frac{B(0,T)}{B(0,t)} \exp \left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T) \right],$$

where $B(0,t), B(0,T)$ are the bond prices as observed at time 0 for durations t, T,

$$\bar{X}(t,T) := \frac{1 - \exp(-g(T-t))}{g},$$

$$\bar{Y}(t,T) := \frac{\Sigma^2}{4g} \left[1 - \exp(-2gt) \right] \bar{X}^2(t,T),$$

and the difference between the time-t actual and forward rate $R(0,t)$:

$$K(t) := r(t) - R(0,t)$$

is the *financial risk factor*, akin to the demographic factor $I_i(t)$. As in the longevity case, the financial risk factor is the difference between actual and forecasted rates for time t, where the forecast is done at time 0. It is the only source of randomness which affects bonds. It is clear that - for any maturity T - the bond value at any point in time $t > 0$ is random. Values at time $t = 0$ only are known. Bond values at $t > 0$ are subject to changes in case the future instantaneous rate at time $t, r(t)$, is different from its forecast today, the forward rate.

1.3 Annuities

Consider an annuity issued on an individual of generation i, aged x at t. Make the annuity payment per period equal to one, for the sake of simplicity, and assume that the annuity is fairly priced and reserved. Since we assumed that financial and demographic risks (Brownian motions) are independent, the cash flow of the annuity at tenor T has a fair value at time t equal to the product of the survival probability[5] S and the discount factor B:

$$S_i(t,T)B(t,T)$$

The whole annuity - which lasts until the extreme age ω - is worth

$$V_i^A(t) = \sum_{T=t+1}^{\omega - x} S_i(t,T)B(t,T)$$

[4]The short-rate process is given directly under the risk-neutral measure, so that no assumption on the market price of financial risk is needed. The parameters of the interest-rate market will be calibrated accordingly.

[5]We implicitly assume that there is no price for demographic risk, so that expectations of functionals of the intensity - such as the survival probability - under the historical measure $\mathbb{P}$ and the risk-neutral one/ones $\mathbb{Q}$ coincide.

at time $t \geq 0$.

2 Risk measurement

2.1 Demographic risk measurement and transfer

With no risk transfer, the fund would incur demographic risk, since at any point in time t the fair value and reserve $V_i^A(t)$ can change because the intensity process does. It can be shown (see Luciano, Regis, Vigna (2012a)) that such change can be approximated up to the second order as follows:

$$\Delta V_i^{AM}(t) = \Delta_A^M(t)\Delta I_i(t) + \frac{1}{2}\Gamma_A^M(t)\Delta I_i^2(t), \tag{6}$$

where the Deltas and Gammas are

$$\Delta_A^M(t) = -\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u) < 0,$$

$$\Gamma_A^M(t) = \sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)[X_i(t,u)]^2 > 0.$$

Their signs show that the annuity value is decreasing and convex in the risk factor.

From now on, we take the point of view of a pension fund which issued such contract at a price $P \geq V_i^A(0)$ and can

- either run into demographic risk, evaluated at its first order impact $\Delta_A^M(t)\Delta I_i(t)$, or
- transfer the risk to a reinsurer or to a special purpose vehicle.

We assume that - when risk is transferred to the reinsurer - the latter covers it using short death contracts in his portfolio, i.e. death contracts he issued or absorbed from insurers. This is the so-called natural hedging, which is likely to be feasible for reinsurers, given the diversification of their portfolios. It is instead difficult to envisage pension funds who are short (issued) death contracts, i.e. act as insurers.

We ask ourselves at what fair price the reinsurer can absorb the demographic risk of the annuity. To this end, we assume that coverage of risk is done by the reinsurer up to first-order changes. He Delta-covers risk[6] by using a position in

[6] We maintain the assumption of delta - as opposite to delta-gamma - coverage for all risks below. In principle, going from delta to delta-gamma coverage just requires the use of additional death contracts and the introduction of more equations. No major conceptual difference seems to be at stake. For this reason, we disregard the extension in the whole paper.

$\mathbb{N}$ death contracts on individuals of the same generation, gender and age. At time t, a death contract which covers the period (t, T) is priced

$$V_i^D(t,T) = \sum_{u=t+1}^{T} B(t,u)\left[S(t,u-1) - S(t,u)\right].$$

and is affected by a change in mortality intensity ΔI_i as follows:

$$\Delta V_i^D(t,T) = \Delta_D^M(t,T)\Delta I_i(t) + \frac{1}{2}\Gamma_D^M(t,T)\Delta I_i^2(t)$$

where

$$\begin{aligned}
\Delta_D^M(t,T) &= \sum_{u=t+1}^{T} B(t,u)\left[-S_i(t,u-1)X_i(t,u-1) + S_i(t,u)X_i(t,u)\right] > 0, \\
\Gamma_D^M(t,T) &= \sum_{u=t+1}^{T} B(t,u)\left[S_i(t,u-1)X_i^2(t,u-1) - S_i(t,u)X_i^2(t,u)\right] < 0.
\end{aligned}$$

Their signs show that the death contract is increasing and concave in the risk factor. The position $\mathbb{N}$ is determined so that the Delta of the portfolio made by the annuity - in which the reinsurer is short, since he took demographic risk in charge - and the death contract is zero:

$$(-\Delta_A^M + \mathbb{N}\Delta_D^M)\Delta I_i(t) = 0$$

$$\mathbb{N}(t,T) = \frac{\Delta_A^M}{\Delta_D^M} =$$

$$= -\frac{\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u)}{\sum_{u=t+1}^{T} B(t,u)\left[-S_i(t,u-1)X_i(t,u-1) + S_i(t,u)X_i(t,u)\right]} < 0.$$

The reinsurer is short the death contract, since the annuity value increases when longevity is greater than forecasted, while the death value decreases. As a consequence, the increase in the payments to annuitants due to an unexpected shock in longevity is compensated by the decrease in the expected payments due to life-insurance policyholders.

The fair cost of such coverage is the value of the death contracts needed for hedging:

$$V(t,T) = -\mathbb{N}\sum_{u=t+1}^{T} B(t,u)\left[S_i(t,u-1) - S_i(t,u)\right] =$$

$$= \frac{\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u)\sum_{u=t+1}^{T} B(t,u)\left[S_i(t,u-1) - S_i(t,u)\right]}{\sum_{u=t+1}^{T} B(t,u)\left[-S_i(t,u-1)X_i(t,u-1) + S_i(t,u)X_i(t,u)\right]}. \quad (7)$$

We assume that - in order to absorb demographic risk - the reinsurer charges the fund with a price C which is not smaller than the fair price, $C \geq V(t,T)$.

2.2 Financial risk measurement

After coverage of demographic risk through the reinsurer, the fund is left with

- an amount $P - C$ to invest in financial assets, i.e. bonds. This becomes P if ever demographic risk is not transferred;
- the financial risk of the annuity. Since the annuity is fairly priced, its value is indeed subject to interest rate fluctuations.

So, it suffers financial risk on assets and liabilities, as follows.

2.2.1 Assets

Any bond which enters the assets of the fund - for the sake of simplicity we consider zero-coupon bonds only - is subject to financial risk. Its sensitivity to changes in K - ΔK - is well known:

$$\Delta_B^F(t,T) = -B(t,T)\bar{X}(t,T) < 0, \tag{8}$$

$$\Gamma_B^F(t,T) = B(t,T)\bar{X}^2(t,T) > 0. \tag{9}$$

Bonds are decreasing and convex in discrepancies between the actual and forecasted interest rates.

2.2.2 Liabilities

Also the annuity value, which enters the liabilities, is subject to financial risk, since it is fairly priced. The effect of a change in K on the annuity is:

$$\Delta V_i^{AF}(t) = +\Delta_A^F(t)\Delta K(t) + \frac{1}{2}\Gamma_A^F(t)\Delta K^2(t),$$

where

$$\Delta_A^F(t) = -\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) < 0,$$

$$\Gamma_A^F(t) = \sum_{u=t+1}^{\omega-x} B(t,u)S(t,u)[\bar{X}(t,u)]^2 > 0.$$

The sign of these coefficients reveals that the annuity is decreasing and convex in discrepancies between the actual and forecasted interest rates, exactly as the bonds are.

We assume that the position in bonds - given the annuity - is chosen so as to maximize the portfolio (assets and liabilities) expected return. We assume that the fund can not borrow resources, and can thus invest up to the amount received as a premium for the annuity, P, net of reinsurance costs (either 0 or C).[7]

[7] Any remaining fund is kept as cash. We assume that cash provides no return and no risk. This is why it does not appear in the computations below.

3 Maximizing returns

At time t, the fund can consider maximizing expected returns, by buying[8] a number n of bonds - with fixed maturity T - which stays between 0 and the ratio of $P-C$ or P to the time$-t$ price of bonds with maturity T. The fund can indeed invest in assets either $P-C$, if he transferred demographic risk, or P, if he did not. As a result, the fund has a portfolio made up by the annuity (short) and n long bonds, whose instantaneous expected return μ is

$$\mu = \mathbb{E}_t\left[-V_A^F(t+dt) + V_A^F(t) + n\left[B(t+dt,T) - B(t,T)\right]\right]$$

Since only the second part depends on n, the fund chooses this number as high as possible if $\mathbb{E}_t B(t+dt,T) > B(t,T)$, equal to zero in the opposite case. Notice that this expected return – and all the expectations and variances from now on – is computed under the historical measure $\mathbb{P}$. The dynamics of the short rate under $\mathbb{P}$ is

$$dr(t) = (g\theta - (g+\lambda\sigma)r(t))dt + \Sigma dW_F^{'}(t), \tag{10}$$

where $\theta, g > 0, \Sigma > 0$, λ is the market price of risk, which we assume constant and $W_F^{'}$ is a Brownian motion under the historical measure. Under the Hull-White return hypotheses - and using first order approximations for returns over the time interval Δt - this condition is verified if and only if

$$\mathbb{E}_t[K(t+\Delta t) - K(t)]\Delta_B^F(t,T) > 0 \tag{11}$$

i.e.

$$\mathbb{E}_t[K(t+\Delta t)] < 0 \tag{12}$$

since Δ_B^F is always negative and $K(t) = r(t) - R(t,t) = r(t) - r(t) = 0$. The expected value of $K(t+\Delta t)$ under $\mathbb{P}$ is[9]

$$\begin{aligned}
\mathbb{E}_t[K(t+\Delta t)] &= r(t)e^{-(g+\lambda\Sigma)(\Delta t)} + \frac{\theta g}{g+\lambda\Sigma}\left[1 - e^{-(g+\lambda\Sigma)\Delta t}\right] + \\
&-\left(\theta - \frac{\Sigma^2}{2g^2}\right)(1-e^{-g\Delta t}) - \frac{\Sigma^2}{2g^2}(1-e^{-g\Delta t})e^{-g} - e^{-g\Delta t}r(t) = \\
&= r(t)e^{-g\Delta t}\left(e^{-(\lambda\Sigma)\Delta t} - 1\right) + \theta\left(\frac{g}{g+\lambda\Sigma} - 1\right) + \\
&+\theta e^{-g\Delta t}\left(1 - \frac{ge^{-\lambda\Sigma\Delta t}}{g+\lambda\Sigma}\right) + \frac{\Sigma^2}{2g^2}\left(1-e^{-g\Delta t}\right)^2.
\end{aligned}$$

[8]This means that we do not want the fund neither to borrow cash in order to invest in bonds, nor vice versa (issue bonds to hold cash). This constraint can obviously be relaxed.

[9]The variance of the financial risk factor K is

$$\mathbb{V}ar_t\left[K(t+\Delta t)\right] = \frac{\Sigma^2}{2(g+\lambda\Sigma)}\left(1 - e^{-2(g+\lambda\Sigma)\Delta t}\right).$$

It follows that the optimal position on the bonds is

$$n^* = \begin{cases} (P-C)/B(t,T) & \text{if (12) holds true and demographic risk was transferred} \\ P/B(t,T) & \text{if (12) holds true and demographic risk was not transferred} \\ 0 & \text{otherwise} \end{cases}$$

The financial risk incurred by the fund, as a consequence of this asset policy, can be evaluated at first order as follows. It is

$$\left[-\Delta V_A^F(t) + \frac{P-C}{B(t,T)}\Delta_B^F(t,T)\right]\Delta K(t) =$$

$$= \left[-\Delta_A^F(t) - \frac{P-C}{B(t,T)}B(t,T)\bar{X}(t,T)\right]\Delta K(t)$$

$$= \left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - (P-C)\,\bar{X}(t,T)\right]\Delta K(t) \qquad (13)$$

if $n^* = (P-C)/B(t,T)$. It has the same expression with $C = 0$ if $n^* = P/B(t,T)$. It is

$$-\Delta V_A^F(t)\Delta K(t) = -\Delta_A^F(t)\Delta K(t) =$$

$$= \Delta K(t)\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) \qquad (14)$$

if $n^* = 0$.

So, if the fund maximizes expected returns, it will be exposed to either (13) - at most with $C = 0$ - or (14) - according to the optimal policy n^*.
At the same time, if the fund decided not to transfer demographic risk - with the usual convention of approximating exposures to the first order - he will be exposed to a longevity risk of

$$\Delta I_i(t)\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u)$$

Summing up, one of the following situations can arise:

- the fund has demographic and financial risk. The latter arises from liabilities only, since $n^* = 0$, i.e. it turned out to be optimal to keep the assets in cash. Risks are

$$\Delta I_i(t)\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u)$$

$$\Delta K(t)\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) \qquad (15)$$

  while maximized expected returns equal

$$\mathbb{E}_t\left[-V_A^F(t+dt) + V_A^F(t)\right] \simeq \mathbb{E}_t\left[\Delta K(t)\right]\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) \quad (16)$$

- the fund has demographic and financial risk from assets (bonds) and liabilities, since $n^* = P/B$. Risks are

$$\Delta I_i(t) \sum_{u=t+t+1}^{\omega-x} B(t,u) S_i(t,u) X_i(t,u)$$

$$\Delta K(t) \left[\sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) - P\bar{X}(t,T) \right] \tag{17}$$

while maximized expected returns equal

$$\mathbb{E}_t \left[-V_A^F(t+dt) + V_A^F(t) + \frac{P}{B(t,T)} \left[B(t+dt,T) - B(t,T) \right] \right] \tag{18}$$

$$\simeq \mathbb{E}_t \left[\Delta K(t) \right] \left[\sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) - P\bar{X}(t,T) \right] \tag{19}$$

- the fund has no demographic risk, since he transferred it, and has financial risk from liabilities only, since $n^* = 0$

$$\Delta K(t) \sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) \tag{20}$$

His maximized expected returns equal

$$\mathbb{E}_t \left[-V_A^F(t+dt) + V_A^F(t) \right] \tag{21}$$

$$\simeq \quad \mathbb{E}_t \left[\Delta K(t) \right] \sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) \tag{22}$$

- the fund has no demographic risk - which was transferred - and has financial risk from assets, since $n^* = (P-C)/B$, and liabilities. This risk is equal to

$$\Delta K(t) \left[\sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) - (P-C)\, \bar{X}(t,T) \right] \tag{23}$$

Its maximized expected returns equal

$$\mathbb{E}_t \left[-V_A^F(t+dt) + V_A^F(t) + \frac{P-C}{B(t,T)} \left[B(t+dt,T) - B(t,T) \right] \right] \tag{24}$$

$$\simeq \quad \mathbb{E}_t \left[\Delta K(t) \right] \left[\sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) - (P-C)\, \bar{X}(t,T) \right] \tag{25}$$

4 Trade-offs

In order to study the risk-return trade-offs inherent in the four strategies resulting from expected return maximization, let us introduce the following notation:

$$
\begin{aligned}
\alpha &:= \sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) X_i(t,u) > 0 \\
\beta &:= \sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) > 0 \\
\gamma &:= \beta - P\bar{X} < \beta \\
\delta &:= \gamma + C\bar{X} > \gamma
\end{aligned}
$$

Let us denote as C^* the amount paid for demographic risk transfer. Depending on the fund's choice, we may have $C^* = C$ or $C^* = 0$. The four expected-return-maximizing strategies are described in Table 1 by the asset allocation strategies n^* and demographic-risk transfer-payment C^*, together with the *risks and net expected returns* they entail. Returns are net of the costs C^* of demographic-risk transfer. So, for strategies 1 and 2 α is the Delta of the portfolio with respect to mortality risk, while β, γ and δ are the Deltas of the portfolios for the four strategies with respect to financial risk.

Strategy	n^*	C^*	**Dem risk**	**Fin risk**	**Net expected return**
1	0	0	$\alpha\Delta I$	$\beta\Delta K$	$\beta\mathbb{E}[\Delta K]$
2	P/B	0	$\alpha\Delta I$	$\gamma\Delta K$	$\gamma\mathbb{E}[\Delta K]$
3	0	C	0	$\beta\Delta K$	$\beta\mathbb{E}[\Delta K] - C$
4	(P-C)/B	C	0	$\delta\Delta K$	$\delta\mathbb{E}[\Delta K] - C$

Table 1: Risks and expected return

Notice that we subtract the whole cost of reinsurance – which lasts for the whole annuity maturity, $\omega - x$ – even though financial returns are computed over the next period Δt. This will make our expected returns low. [10]. Each strategy entails a cash flow, i.e. a *liquidity* injection or outflow, equal to $P - C^* - n^* B$. In order to compare the strategies we cannot use a standard risk/return frontier, for at least two reasons. First, we have not measured risk with any synthetic measure, such as the variance. We have measured it through the change in the portfolio value - i.e. through the profit/loss - corresponding to every specific difference between forecasted and actual mortality ΔI or interest rate ΔK. We have a change in portfolio value for every couple or scenario (ΔI, ΔK). Second, we have two sources of risk, demographic and financial. As a consequence, we would need a risk/return frontier in three dimensions. In order to reconstruct a

[10] The model can accomodate any splitting of the reinsurance cost over the maturity of the annuity

risk/return trade-off, without loosing the two sources of risk, in the next Section 4.1 we proceed in three steps. We first recognize the link between the scenario-based risk representation and a Value-at-Risk (VaR) risk measurement for each risk factor. The second step consists in passing from the VaR of the factor to the VaR of the portfolio strategy. The third steps consists in summing up the VaRs due to financial and demographic risk to obtain the Overall VaR.

4.1 One-standard deviation shocks and VaRs

Observe first that the expected values of the risk factors changes, $\Delta I_i = \Delta I_i(t+\Delta t)$ and $\Delta K = \Delta K(t+\Delta t)$ are equal to the expected values of the mortality intensity and interest rate, $\lambda_i(t+\Delta t))$ and $r(t+\Delta t)$, which we computed above, in (1) and (4), net of the corresponding forward rate. The variances are the ones computed in (2) and (5). So, using (1), (4), (2) and (5), we can compute $\mathbb{E}[\Delta I_i], \mathbb{V}ar[\Delta I_i], \mathbb{E}[\Delta K], \mathbb{V}ar[\Delta K]$.

A synthetic way to compare the four alternative strategies above consists in looking at what happens if both the mortality and interest rate over the next instant suffer a shock, i.e. have a realization which is not equal to their mean, $\mathbb{E}[\Delta I_i], \mathbb{E}[\Delta K]$. Each move $(\Delta I_i, \Delta K)$ different from $\mathbb{E}[\Delta I_i], \mathbb{E}[\Delta K]$, once substituted in the risk/return triples, provides a *scenario-based assessment of the strategy* itself. Consider first a positive or negative one-standard-deviation shock on the longevity of generation i and on interest rates:

$$\Delta I_i = \mathbb{E}[\Delta I_i] \pm 1 \times \sqrt{\mathbb{V}ar[\Delta I_i]} \tag{26}$$

$$\Delta K = \mathbb{E}[\Delta K] \pm 1 \times \sqrt{\mathbb{V}ar[\Delta K]} \tag{27}$$

It is straightforward to construct from this an evaluation of the VaR-type. Since both the intensity and the interest rate are Gaussian, indeed, looking at a one-standard-deviation shock means to examine the worst occurrence for I and K in 84% or 16% of the cases. Expressions (26) and (27) give the *VaR of the risk factors* at the level of confidence 84% - if we take $-1 \times \sqrt{\mathbb{V}ar[\Delta I_i]}$ - and 16%, if we take $+1 \times \sqrt{\mathbb{V}ar[\Delta I_i]}$. By changing the number of standard deviations examined - bringing it from 1 to 1.65 or 2.33, for instance - we would be looking at the worst scenarios for I and K in 95% and 5% or 99% and 1% of the cases. In general, we can fix a confidence level $1-\epsilon$ (say 99, 95, 84) or ϵ at which the VaR of the risk factors can be evaluated, by choosing appropriately the constant in front of the standard deviation. Let $n(\epsilon)$ be that constant. So, the VaR of the two risk factors at the confidence level $1-\epsilon$ is

$$VaR_{1-\epsilon}(\Delta I) = \mathbb{E}[\Delta I_i] - n(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]} \tag{28}$$

$$VaR_{1-\epsilon}(\Delta K) = \mathbb{E}[\Delta K] - n(\epsilon)\sqrt{\mathbb{V}ar[\Delta K]} \tag{29}$$

However, in the end we are interested in the *VaR of the portfolio*, not in the VaR of the risk factors. According to Table 1, the realizations of the portfolio gain/loss are of the type $k\Delta I_i$ or $k\Delta K$, where the constant k can be either positive or negative ($k = \alpha, \beta, \gamma, \delta$). To an increase in the risk factor corresponds a portfolio loss if $k < 0$, a gain if $k > 0$. This means that the VaR of the portfolio due to demographic risk, is

$$\begin{aligned} kVaR_{1-\epsilon}(\Delta I_i) &= k\left[\mathbb{E}[\Delta I_i] - n(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]}\right] \qquad \text{if } k > 0 \qquad (30)\\ kVaR_{\epsilon}(\Delta I_i) &= k\left[\mathbb{E}[\Delta I_i] + n(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]}\right] \qquad \text{if } k < 0 \qquad (31)\end{aligned}$$

Similarly for interest-rate risk. Table 2 reports the values of the financial and demographic VaR for each strategy.

Table 2: Demographic and Financial VaR for the four strategies

	Contribution to the strategy VaR	
Strategy	Demographic risk	Financial Risk
1	$\alpha VaR_{1-\epsilon}(\Delta I_i)$	$\beta VaR_{1-\epsilon}(\Delta K)$
2	$\alpha VaR_{1-\epsilon}(\Delta I_i)$ $\alpha VaR_{1-\epsilon}(\Delta I_i)$	$\gamma VaR_{1-\epsilon}(\Delta K)$ if $\gamma > 0$ $\gamma VaR_{\epsilon}(\Delta K)$ if $\gamma < 0$
3	0	$\beta VaR_{1-\epsilon}(\Delta K)$
4	0 0	$\delta VaR_{1-\epsilon}(\Delta K)$ if $\delta > 0$ $\delta VaR_{\epsilon}(\Delta K)$ if $\delta < 0$

Due to independency between financial and actuarial risk sources, if we sum up the appropriate scenario-based risks or VaRs (where appropriate stands for "based on the need of selecting VaR_ϵ versus $VaR_{1-\epsilon}$") we obtain the strategy-VaR due to both sources of risk. Consider for instance the first strategy, which has risks $(\alpha\Delta I_i, \beta\Delta K)$. Since both coefficients α and β are positive, the VaR of the strategy is

$$\alpha VaR_{1-\epsilon}(\Delta I_i) + \beta VaR_{1-\epsilon}(\Delta K)$$

If we apply a similar reasoning for the other strategies, for each one we can compute the *overall VaR*, which we report in Table 3 together with the strategy's net expected return. This opens the way to representing the trade-offs of the strategies in a familiar way, by associating to each strategy a point in the plane (Overall-VaR, net expected return).

Strategy	**(VaR,expected return) combination**
1	$(\alpha VaR_{1-\epsilon}(\Delta I_i) + \beta VaR_{1-\epsilon}(\Delta K), \beta\mathbb{E}[\Delta K])$
2	$(\alpha VaR_{1-\epsilon}(\Delta I_i) + \gamma VaR_{1-\epsilon}(\Delta K), \gamma\mathbb{E}[\Delta K])$ if $\gamma > 0$ $(\alpha VaR_{1-\epsilon}(\Delta I_i) + \gamma VaR_{\epsilon}(\Delta K), \gamma\mathbb{E}[\Delta K])$ if $\gamma < 0$
3	$(\beta VaR_{1-\epsilon}(\Delta K), \beta\mathbb{E}[\Delta K] - C)$
4	$(\delta VaR_{1-\epsilon}(\Delta K), \delta\mathbb{E}[\Delta K] - C)$ if $\delta > 0$ $(\delta VaR_{\epsilon}(\Delta K), \delta\mathbb{E}[\Delta K] - C)$ if $\delta < 0$

Table 3: Overall VaR for the strategies

5 Comparing strategies

In order to state whether demographic risk transfer is worthwhile, we need to compare specific couples of the strategies listed in Table 1. Indeed, two alternative couples of strategies can be implemented by the fund: either 1 and 3 or 2 and 4. If condition (12) is not met and the optimal number of bonds turns out to be zero ($n^* = 0$), the choice is between strategies 1 and 3, namely

$$\begin{array}{c}(\alpha \Delta I, \beta \Delta K, \beta \mathbb{E}\left[\Delta K\right]) \\ (0, \beta \Delta K, \beta \mathbb{E}\left[\Delta K\right] - C)\end{array}$$

Otherwise, if condition (12) is met and the optimal number of bonds is positive ($n^* > 0$) the choice is between strategies 2 and 4, namely

$$\begin{array}{c}(\alpha \Delta I, \gamma \Delta K, \gamma \mathbb{E}\left[\Delta K\right]) \\ (0, \delta \Delta K, \delta \mathbb{E}\left[\Delta K\right] - C)\end{array}$$

We compare the competing strategies in the plane "Overall-VaR, net expected return". Hence, the comparison is between

$$\begin{array}{l}(\alpha VaR_{1-\epsilon}(\Delta I_i) + \beta VaR_{1-\epsilon}(\Delta K), \beta \mathbb{E}\left[\Delta K\right]) \\ (\beta VaR_{1-\epsilon}(\Delta K), \beta \mathbb{E}\left[\Delta K\right] - C)\end{array}$$

for strategies 1 and 3,

$$\begin{cases} (\alpha VaR_{1-\epsilon}(\Delta I_i) + \gamma VaR_{1-\epsilon}(\Delta K), \gamma \mathbb{E}\left[\Delta K\right]) & \text{if } \gamma > 0 \\ (\alpha VaR_{1-\epsilon}(\Delta I_i) + \gamma VaR_{\epsilon}(\Delta K), \gamma \mathbb{E}\left[\Delta K\right]) & \text{if } \gamma < 0 \end{cases}$$

$$\begin{cases} (\delta VaR_{1-\epsilon}(\Delta K), \delta \mathbb{E}\left[\Delta K\right] - C) & \text{if } \delta > 0 \\ (\delta VaR_{\epsilon}(\Delta K), \delta \mathbb{E}\left[\Delta K\right] - C) & \text{if } \delta < 0 \end{cases}$$

for strategies 2 and 4.
Fixing the confidence level ϵ for the VaR, we can represent the combination of risk and expected return for each strategy on a two-dimensional plot. Actually, the fund could reinsure just a part of its liabilities against longevity risk, by choosing $C^* = \eta C, \eta \in [0, 1]$. In practice, the fund can implement all the linear combinations of the two alternative strategies 1 and 3 or 2 and 4. It is then possible to the represent the set of all the possible strategies with a line that goes from 1 to 3 or from 2 to 4. When $n^* = 0$ the set of possible strategies is characterized by the straight line that crosses 1 and 3 (Figure 1). When instead condition (12) is met, the set of return maximizing strategies for different values of η is represented by a broken line between 2 and 4 (Figure 2). In this case indeed there is no liquidity left, since the fund invests all its available resources in the bond. The kink of the line corresponds lies in the point at which the Delta of the portfolio of assets and liabilities – i.e. short the annuity and long

the bond – with respect to the financial risk is null. If the risk-return preferences of the fund can be described through a utility function on the plane (Expected Financial Return, Overall VaR), the best strategy is identified as the point of the straight line that crosses the highest possible indifference curve, which is also represented in the figures.

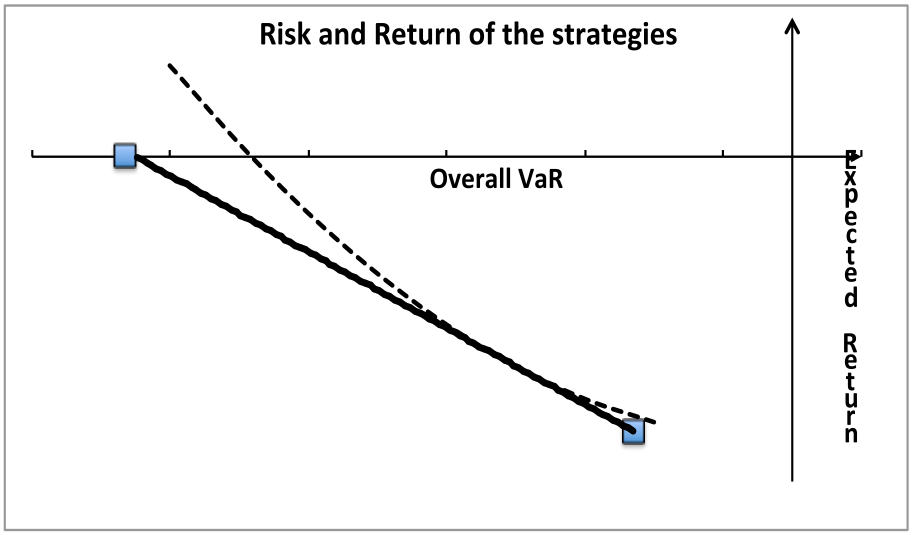

Figure 1: This figure shows the risk-return combinations of the set of strategies, for all the possible values of η when $n^* = 0$. The strategies are represented by the black solid line. The dotted curve is the highest indifference curve which is tangent to the set of strategies. The optimal strategy lies at the intersection between the curve and the line. On the horizontal axis the VaR at a certain level ϵ is reported, while the expected financial return net of reinsurance costs lies on the vertical axis.

In order to appreciate the trade/offs so elicited and the different informations we can convey on them, we introduce and comment an example calibrated on UK mortality and financial data.

6 Implementing and comparing strategies on UK data

We implement the strategies designed above and study their trade-offs using data from the UK market. To be specific, we consider a whole-life annuity sold on a UK male head aged 65 at strategy inception, December 30, 2010; we take financial data from the UK Government market on the same date. We then

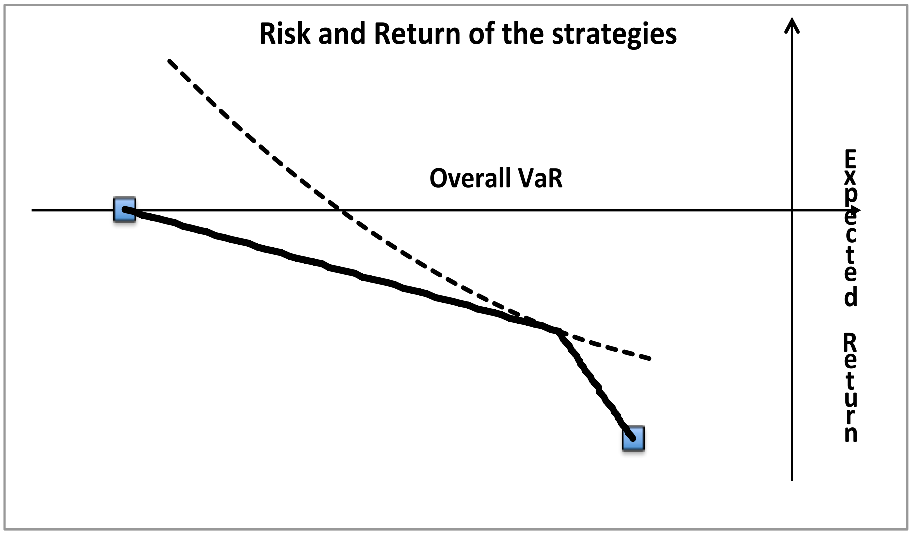

Figure 2: This figure shows the risk-return combinations of the set of strategies, for all the possible values of η, when $n^* > 0$. The strategies are represented by the black broken line. The dotted curve is the highest indifference curve which is tangent to the set of strategies. The optimal strategy lies at the intersection between the curve and the line. On the horizontal axis the VaR at a certain level ϵ is reported, while the expected financial return net of reinsurance costs lies on the vertical axis.

presume that the revenues from annuity sales are invested in UK Government bonds. The Hull-White model is then calibrated under the risk-neutral measure and its parameters are $g = 2.72\%$, $\theta = 17.23\%$, $\Sigma = 0.65\%$, while the initial value of the short-rate is $r(0) = 0.42\%$. The market price of risk λ is chosen so that the long-run mean under the historical measure is around 4%. The survival rates are calibrated from the historical IML tables. For the generation we considered, the model parameters are $a_i = 10.94\%, \sigma_i = 0.07\%$ and $\lambda_i(0) = 0.885\%$. Table 4 summarizes all the relevant parameters.

Table 5 reports the prices and the Deltas of the instruments we use in the example.

The fair price of the annuity – which is also its selling price – is $V^A = P = 13.09$.[11] Being short the annuity, which has exposures $\Delta_A^M = -323.48$ and $\Delta_A^F = -100.92$ the fund remains exposed positively to both risk factors change. The fund operates on the financial market using a 10-year bond, which is priced $B(0, 10) = 0.75576$ and has $\Delta_B^F = -6.617$. We evaluate the hedging strategies we described above at an horizon $\Delta t = 1$ year. The longevity risk factor $I(1)$ is instead expected to be positive, $\mathbb{E}_0[I(1)] = 2.73 * 10^{-7}$ while its variance is $\mathbb{V}ar_0[I(1)] = 5.47 * 10^{-7}$. Demographic risk can be transferred to a reinsurer at

[11] In the computation, we considered $\omega = 110$ years.

Table 4: Calibrated parameters

Symbol	Value
Financial risk	
g	2.72%
Σ	0.65%
θ	17.23%
r(0)	0.42%
Demographic risk	
a_i	10.94%
σ_i	0.07%
$\lambda_i(0)$	0.885%

Table 5: Risk exposures and prices of instruments

Figure	Symbol	Value
Annuity		
Price	V^A	13.09
Exposure to longevity risk	Δ_A^M	-323.48
Exposure to financial risk	Δ_A^F	-100.92
10-year bond		
Price	B(0,10)	0.75576
Exposure to financial risk	Δ_B^F	-6.617

its fair price $C = 2.95$. The expected value of the financial risk factor under the historical measure is slightly negative, equal to $\mathbb{E}_0[K(1)] = -0.04\%$, while its variance is $\mathbb{V}ar_0[K(1)] = 3.79 * 10^{-5}$. As a result of charging expected returns with the whole reinsurance cost $\mathbb{E}_0[K(1)] - C$ will be strongly negative. As remarked in Section 4, we could split C over the life of the annuity.
Since condition (12) is met, the fund maximizes returns by buying as many bonds as possible, in order to reduce its exposure with respect to financial risk. The two strategies it can implement are then 2 and 4, depending on the fund's intention to cover against demographic risk. The coefficient δ for strategy 4 is positive, 12.14, while γ which appears in strategy 4 is negative, -13.69. In Table 6 we report the exposures, the expected financial return net of the reinsurance cost and the remaining liquidity of the two strategies.

Both strategies make use of all the available resources P received as a premium for the annuity sale and no liquidity is left.

Strategy 2 invests all P in $n^* = 17.32$ bonds. It offers a positive expected return, 0.006, since the fund has negative exposure to financial risk: $\Delta^F = -13.69$. The overall VaR, which is computed at a one-year horizon at a 99.9% confidence level, is -0.96 and it is due mostly to longevity risk (-0.76). Strategy

Table 6: Risk exposures, VaR and expected returns

Figure		Strategy	
	Symbol	2	4
Optimal number of bonds	n^*	17.32	13.41
Optimal cost of reinsurance	C^*	0	2.95
Exposure to longevity risk	$\alpha/0$	323.48	0
Exposure to financial risk	γ/δ	-13.69	12.14
Liquidity	$P - C^* - n^*B$	0	0
Expected financial return	μ	0.007	-2.95
$VaR_{99.9\%}$ demographic risk	$VaR_{99.9\%}(\Delta I)$	-0.71	0
$VaR_{99.9\%}$ financial risk	$VaR_{99.9\%}(\Delta K)$	-0.25	-0.23
Overall $VaR_{99.9\%}$	$VaR_{99.9\%}$	-0.96	-0.23
Overall $VaR_{84\%}$	$VaR_{84\%}$	-0.31	-0.08

4 hedges against longevity risk and invests the remaining resources $P - C$ to buy $n^* = 13.42$ bonds. The overall VaR is reduced from -0.96 of strategy 2 to -0.23 on the one side, but expected financial returns are lower (-2.956 vs. 0.007), mainly because the price of 2.95 is paid for longevity risk reinsurance.

Figure 3 represents the trade-off between risk (measured by overall VaR) and expected return in place between these two competing strategies. The fund accepts to deal with a higher risk to obtain a higher expected return if it chooses to implement strategy 2 over 4.

If we specify a utility function for the fund, defined with respect to expected returns and overall VaR, the fund can optimally choose between the competing strategies the strategy that maximizes utility. Let us consider for example a simple expected utility function

$$U(\mu, VaR) = \mu - \xi(VaR_{99.9\%})^2,$$

in which $\xi > 0$ is a measure of risk aversion correlated with the risk aversion coefficient.

Let us set it to $\xi = 5$. Comparing the utility of the two strategies, the fund, which is highly risk averse would choose strategy 4, for which U = -3.22, over strategy 2, for which U = - 4.63. We now allow the fund to reinsure a part $\eta \in [0, 1]$ of its longevity exposure in order to maximize its utility. Figure 4 represents the set of possible strategies, which is the line between 2 and 4, and the highest indifference curve that crosses this set. The tangency point determines the optimal fund strategy. In the figure, the dotted line represents the highest possible indifference curve which is tangent to the set of possible fund strategies. The optimal strategy for this fund implies reinsuring $\eta = 52.99\%$ of the longevity exposure (at a cost of 1.56) and buying 15.25 bonds. Notice that this optimal strategy has no exposure to financial risk. The optimal strategy is

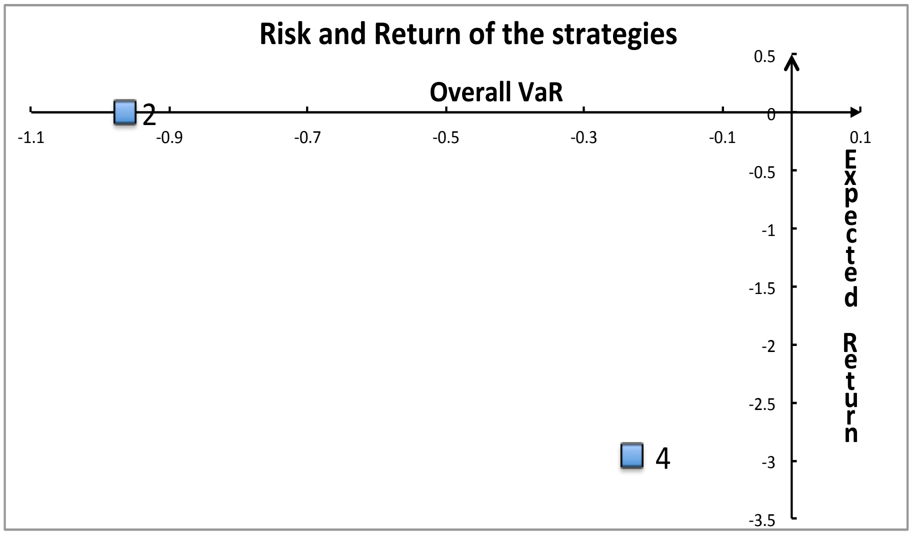

Figure 3: This figure shows the risk-return combinations of strategies 2 and 4. On the horizontal axis the VaR at level 99.9% is reported, while the expected financial return net of reinsurance costs lies on the vertical axis.

then characterized by

$$U^* = -2.13$$

which is way higher than the one of strategy 4, a

$$VaR = -0.34$$

and an instantaneous expected return $\mu = -1.563$, which includes a reinsurance cost of 1.56.

7 Conclusions and related research

This paper explores the overall risk-return trade-off of a pension fund when it is possible to transfer longevity risk and the fund chooses optimally its asset allocation. We measured this trade-off in terms of risk/return combinations and we assessed risk through Value-at-risk from both financial and longevity shocks. We succeeded in quantifying the trade/off and we represented it in the plane expected return-VaR. The optimal transfer choices of the fund are located along the corresponding frontier. We provided a fully calibrated example in which partial retention may be optimal.

Our paper differs from most of the literature in pension fund's management, in which either the mortality of beneficiaries is not modelled directly – see for instance Josa-Fombellida and Rincon-Zapatero (2010) – or the intensity is deterministic (Hainaut and Devolder (2007)). As a consequence, demographic risk

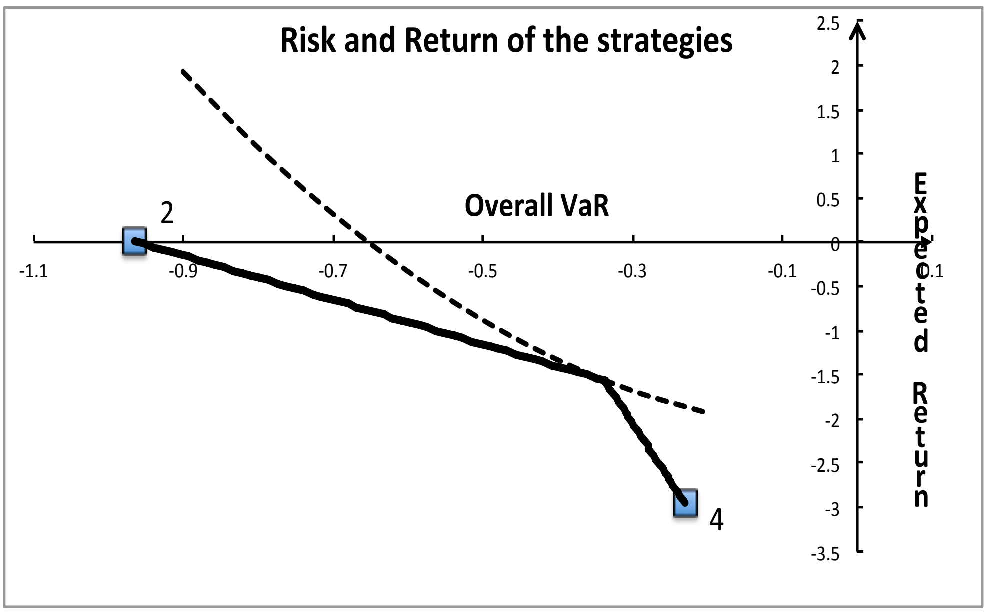

Figure 4: This figure shows the risk-return combinations of the set of strategies (the broken line between 2 and 4) once partial longevity reinsurance is admitted. On the horizontal axis the VaR at level 99.9% is reported, while the expected financial return net of reinsurance costs lies on the vertical axis. The dotted line represents the highest possible indifference curve of the utility function that crosses the set of strategies.

transfer is not examined since the longevity risk issue is not tackled. It differs also from the flourishing literature on the possibility of transferring demographic risk through the financial market, using s-forward, q-forwards or other over-the-counter products. For a review on these alternative transfer possibilities, we address the reader to Biffis and Blake (2010a). That literature indeed focuses on ways to alleviate the cost of the transfer, or make it fairly priced, not on the optimal retention.

A paper which addresses the optimal retention, but is centered on asymmetric information about longevity risk, is Biffis and Blake (2010b). In their case the optimal retention does not come from a risk/return trade-off, but aims at trading-off the overpricing due to low information of the longevity buyer (when retention is low) and the cost of capital (when retention is high).

Our analysis could be extended to the case in which the fund is subject to regulatory capital requirements. If we link – as we could easily do – our VaR measure to the capital requirement of the fund, the capital absorption and its cost enter our picture in quite a straightforward way.

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