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Efficient versus inefficient hedging strategies in the presence of financial and longevity (value at) risk*

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Abstract

This paper provides a closed-form Value-at-Risk (VaR) for the net exposure of an annuity provider, taking into account both mortality and interest-rate risk, on both assets and liabilities. It builds a classical risk-return frontier and shows that hedging strategies - such as the transfer of longevity risk - may increase the overall risk while decreasing expected returns, thus resulting in inefficient outcomes. Once calibrated to the 2010 UK longevity and bond market, the model gives conditions under which hedging policies become inefficient.

JEL Classification: G22, G32.

1 Introduction

Longevity risk - which is the risk of unexpected improvements in survivorship - is known to be an important threat to the safety of annuity providers, such as pension funds. These institutions run the risk of seeing their liabilities increase over time, when the actual survival rate of their members is greater than the forecasted one. As of 2007, the exposure of pension funds and other annuity providers to unexpected improvements in life expectancy has been quantified in 400 billion USD for the US and UK, more than 20 trillion USD worldwide (see

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Biffis and Blake (2010)). Annuity providers are also exposed to financial risks on both assets and liabilities, as soon as the latter are fairly evaluated.

Fair evaluation may be justified on purely economic grounds or may be mandated by accounting rules and regulation. Nowadays, the IASB (International Accounting Standard Board) forces evaluation of liabilities at fair value. Regulatory provisions of the Solvency II type require to align capital standards to the market value of liabilities. Given the current accounting and forthcoming regulatory rules, then, it is important to evaluate the effect of interest-rate risk on both assets and liabilities. Regulatory and accounting interventions make a fair-value based, possibly holistic view of longevity and financial risk matter, since liabilities are subject to both, even when assets are subject to financial risk only. As this paper will show, an holistic view permits also to highlight non trivial trade-offs in risk hedging.

Numerical approaches to longevity or financial risk evaluation problem have been the object of many efforts, both in the industry and in the Academia. A number of them have concentrated on Value-at-risk (VaR), since, in spite of its lack of coherence, it is the risk measure incorporated in the current and forthcoming supervisory standards, such as the Solvency II directive. In Insurance, most of the time VaR from market risk, including interest-rate risk, and mortality risk are computed separately. Mortality risk enters into such computations in the form of idiosyncratic or systematic risk. In the former case there is an implicit or explicit assumption that systematic longevity risk has been reinsured away, a circumstance that we rule out here, but which has been studied, together with financial risk, by Hainaut and Devolder (2007) and Battocchio et al. (2007). In the second case idiosyncratic risk is assumed to be negligible, because the portfolio of the insurer is well diversified, while systematic risk is captured by modelling the mortality intensity of annuitants as a stochastic process instead of a (known) deterministic function of age. This is our approach. Farr et al. (2008) provide a detailed survey of the current VaR practices and simulation approaches in insurance, mainly related to economic capital computations. They conclude that "Where stochastic models (for longevity risk) are used, they are typically run as stand-alone models, separately from the modeling of other risks [..] A fully integrated stochastic approach may also be possible, where mortality is modeled together with other risks. This has the advantage of allowing modeling for interactions between mortality and other risks, such as economic risks. However, run times are usually a limiting factor."

This paper aims at filling the gap pointed out by Farr et al. (2008), by obtaining a closed-form expression for VaR of the net exposure (assets minus liabilities) of a simple insurance portfolio, in the presence of interest-rate and demographic risk. This overcomes the problem of run times and permits to study analytically the efficiency versus inefficiency of hedging strategies. Strategies which are a priori expected to reduce the overall riskiness of an insurance portfolio - at the price of a reduction in its expected return - turn out to be able to increase it too. Specific circumstances under which this occurs, i.e. under which the strategy is inefficient, are given, both in theory and in an application to the UK bond and longevity data. In order to obtain closed-form evaluations, we choose a parsimonious continuous-time model for longevity risk, together with a standard model for interest rates. The description of the actuarial and financial market allows us to obtain easy-to-compute analytical expressions for both the expected return and the risk associated to a portfolio of assets and liabilities. For the sake of simplicity - but without loss of generality - we focus on the portfolio of an annuity provider, such as a pension fund.

The paper is structured as follows. Section 2 formalizes our set up for both longevity and financial risk. Section 3 measures their effects on the net exposure of the annuity provider. Section 4 introduces a VaR measure for the overall risk of assets and liabilities. Section 5 spells out the trade-offs between risk and return, pointing to the existence of efficient and inefficient parts of the frontier (VaR, Expected Return). It describes the different fund strategies along that frontier and how they can be matched with the preferences of the fund. Section 6 provides a calibrated example using financial and demographic data from the UK market. Section 7 concludes.

2 Set up

Let us place ourselves in a standard, continuous-time framework. Consider a time interval $\mathcal{T} = [0, T], T < \infty$, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a multidimensional standard Wiener $W(\omega, t), t \in \mathcal{T}$. The space is endowed with the filtration generated by w, $\mathbf{F}^w = \{\mathcal{F}_t\}$. We adopt a stochastic extension of the classical Gompertz law for mortality description and we stick to the Hull-White model for interest-rate risk, as in Luciano et al. (2012a).

2.1 Demographic risk

Mortality risk - which, with a slight abuse of terminology, we call also longevity or demographic risk - exists since death occurs as a Poisson process, with an intensity which, instead of being deterministic as in the classical actuarial framework, is stochastic. This permits experienced mortality to be different from the forecasted one. At each point in time there is an actual mortality intensity, $\lambda(t)$, which may differ from its forecast at any previous point in time, the forward intensity. If the forecast is done at time 0, we denote it as $f(0, t)$. So, longevity risk arises because the actual intensity $\lambda(t)$ may differ from the forward mortality intensity $f(0, t)$.

This stochastic intensity - or stochastic-mortality - approach, which is by-now quite well known in the literature, has the advantage of making closed-form evaluations, as well as description of age, period and cohort effects in mortality, possible. If the intensity is described by linear affine processes, the survival function is indeed known in closed form and can be calibrated using a limited number of parameters. In order to stay in the linear class and to keep the distinction between age, period and cohort effects, but to be extremely parsimonious, we assume that - under $\mathbb{P}$ - the mortality intensity of a head aged x at calendar time t - which belongs to the generation and gender born at time

$i = t - x$ - is described by a so-called Ornstein-Uhlenbeck process, without mean reversion (OU):

$$d\lambda_i(t) = a_i \lambda_i(t) dt + \sigma_i dW_i(t),$$

where $a_i > 0$, $\sigma_i > 0$, W_i is a standard one-dimensional Brownian motion in W. In the notation we omit the dependence on x, since once calendar time and generation or gender are specified, age is uniquely determined.

This intensity extends - with the inclusion of a diffusive term - the classical Gompertz law

$$d\lambda_i(t) = a_i \lambda_i(t) dt,$$

where $a_i > 0$ is the rate of growth of the force of mortality. Expected intensity increases over age:

$$\mathbb{E}_t(\lambda_i(t + \Delta t)) = \lambda_i(t) \exp(a_i \Delta t) = f_i(t, t + \Delta t) + \frac{\sigma_i^2}{2a_i^2} \left[1 - \exp\left(a_i \Delta t\right)\right]^2, \quad (1)$$

The instantaneous volatility of death intensity is constant, while the overall variance increases exponentially in time:

$$\mathbb{V}ar_t(\lambda_i(t + \Delta t)) = -\frac{\sigma_i^2}{2a_i} \left[1 - \exp\left(2a_i \Delta t\right)\right]. \quad (2)$$

By assuming that there is an intensity process for each generation, we intend to capture longevity discrepancies, especially improvements, over generations. We indeed have (and will calibrate) one drift and one diffusion for each generation (and gender, obviously). This - together with the OU choice - makes the overall mortality model flexible but still parsimonious. Empirical explorations have shown that it fits well actual per-cohort mortality (see Sherris and Wills (2008), among others).

Since it belongs to the affine class, the model provides a closed-form expression for the survival probability of generation i at any point in time t and up to any horizon T. Using a transformation from Jarrow and Turnbull (1994), the survival probability can be written as

$$\begin{aligned} S_i(t, T) &= \mathbb{E}_t \left[\exp\left(-\int_t^T \lambda_i(s) ds \right) \right] = \\ &= \frac{S_i(0, T)}{S_i(0, t)} \exp\left[-X_i(t, T) I_i(t) - Y_i(t, T) \right], \end{aligned}$$

where $S_i(0, T)$ and $S_i(0, t)$ are the survival probabilities at time 0, X_i and Y_i are deterministic functions

$$X_i(t, T) := \frac{\exp(a_i(T - t)) - 1}{a_i},$$

$$Y_i(t, T) := -\frac{\sigma_i^2 \left[1 - \exp\left(2a_i t\right)\right] X_i(t, T)^2}{4a_i},$$

and $I_i(t)$ is the difference between the actual mortality intensity of generation i at t and its forecast at time $0, f_i(0,t)$. We interpret this difference as the *mortality or demographic risk factor*:

$$I_i(t) := \lambda_i(t) - f_i(0,t).$$

It is the discrepancy between realization and forecast which makes the pension fund exposed to mortality risk. Indeed, only the survival probabilities at the current date ($t = 0$) are known, while the probabilities which will be assigned at any future point in time ($t > 0$) are random variables. We will see below that this makes the reserves of the pension fund at any future point in time stochastic, and generates the demographic risk it has to cover. Randomness enters through the factor $I_i(t)$ and affects the whole survival curve, namely $S_i(t,T)$ for every T. From now on, since we focus on a specific generation, we omit the dependence on i.

2.2 Financial risk

In order to seize the effects of interest rate changes on assets and liabilities we need to select a model for financial risk. The natural choice is to assume that interest rates follow an Ornstein-Uhlenbeck (OU) or constant-parameter Hull-and-White one-factor model. This is a standard choice in Financial modelling, able to provide us with closed form formulas for pricing and hedging, parsimonious but flexible enough to be popular in applications. The instantaneous interest rate in the Hull-White model is assumed to have the following dynamics under a measure[1] $\mathbb{Q}$ equivalent to $\mathbb{P}$:

$$dr(t) = g(\theta - r(t))dt + \Sigma dW_F(t), \tag{3}$$

where $\theta, g > 0, \Sigma > 0$ and W_F is a univariate Brownian motion independent[2] of W_i for all i; θ is the long-run mean of the short-rate process, while the parameter g is the speed of mean reversion. As a consequence, the instantaneous rate has expectation and variance equal to

$$\mathbb{E}_t \left[r(t + \Delta t) \right] = r(t)e^{-g\Delta t} + \theta \left[1 - e^{-g\Delta t} \right], \tag{4}$$

$$\mathbb{V}ar_t(r(t + \Delta t)) = \frac{\Sigma^2}{2g} \left[1 - \exp\left(-2g\Delta t\right) \right]. \tag{5}$$

No arbitrage and completeness hold in the financial market. The corresponding zero-coupon bond price - if the bond is evaluated at t and has maturity T -

[1]The short-rate process is given directly under the risk-neutral measure, so that no assumption on the market price of financial risk is needed. The parameters of the interest-rate market will be calibrated accordingly.

[2]Under the original measure we have $W = (W_1, W_2, ...W_N, W_F')$ where N is the maximum number of generations alive in $\mathcal{T}$, while W_F' is the Brownian motion which corresponds to W_F according to Girsanov' theorem. Independency of financial and actuarial risk is assumed under $\mathbb{P}$ and preserved under $\mathbb{Q}$ because of the diffusive nature of uncertainty: see Dhaene et al. (2013).

is

$$B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\right] =$$

$$= \frac{B(0,T)}{B(0,t)} \exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right],$$

where $B(0,t), B(0,T)$ are the bond prices as observed at time 0 for durations $t, T, \bar{X}$ and $\bar{Y}$ are deterministic functions

$$\bar{X}(t,T) := \frac{1 - \exp(-g(T-t))}{g},$$

$$\bar{Y}(t,T) := \frac{\Sigma^2}{4g}\left[1 - \exp(-2gt)\right]\bar{X}^2(t,T),$$

and the difference between the time-t actual and forward rate, denoted as $R(0,t)$:

$$K(t) := r(t) - R(0,t)$$

is the *financial risk factor*, akin to the demographic factor $I(t)$. As in the longevity case, the financial risk factor is the difference between actual and forecasted rates for time t, where the forecast is done at time 0. It is the only source of randomness which affects bonds. It is clear that - for any maturity T - the bond value at any point in time $t > 0$ is random. Values at time $t = 0$ only are known.

3 Risk measurement

This section studies how specific forecast errors in mortality or interest rate impact on the net exposure (assets minus liabilities) of the pension fund, given that

- he can transfer part or the whole longevity risk to a reinsurer
- ha can hedge the financial risk of assets by duration matching.

This section permits to assess the impact of randomly distributed forecast errors. The last task, together with the study of the trade-off between risk and returns, will be the subject of the sections to follow.

3.1 Demographic risk measurement

Consider an annuity issued on an individual of generation i, aged x at t. Make the annuity payment per period equal to one, for the sake of simplicity, and assume that the annuity is fairly priced and reserved. Assume that financial and demographic risks (Brownian motions) are independent and that no risk premium for longevity risk exists[3]. The cash flow of the annuity at tenor T has a fair value at time t equal to the product of the survival probability S and the discount factor B:

$$S_i(t,T)B(t,T)$$

The whole-life annuity - which lasts until the extreme age ω - is worth

$$V_i^A(t) = \sum_{u=t+1}^{\omega-x} S_i(t,u)B(t,u)$$

The fund incurs demographic risk, in the sense that at any point in time t the fair value and reserve $V_i^A(t)$ can change because the intensity process does. Such change can be approximated up to the second order as follows:

$$\Delta V_i^{AM}(t) = \Delta_A^M(t)\Delta I_i(t) + \frac{1}{2}\Gamma_A^M(t)\Delta I_i^2(t), \tag{6}$$

where the Deltas and Gammas are

$$\Delta_A^M(t) = -\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u) < 0,$$

$$\Gamma_A^M(t) = \sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)[X_i(t,u)]^2 > 0.$$

The annuity value is decreasing and convex in the risk factor.

From now on, we take the point of view of a pension fund which issued such contract at a price $P \geq V_i^A(0)$ and can

- either run into demographic risk, evaluated at its first-order impact $\Delta_A^M(t)\Delta I_i(t)$, or
- transfer the risk to a reinsurer, at least partially.

We want to build the risk-return frontier linked to such hedging policy. We assume that - in order to absorb demographic risk - reinsurers charge the fund with a price C which is not smaller than its fair price, determined consistently with the model. To establish the fair price, we assume that - when risk is transferred to the reinsurer - the latter covers it using short death contracts in his portfolio, i.e. death contracts he issued or absorbed from insurers. This is the so-called natural hedging, which is likely to be feasible for reinsurers, given the diversification of their portfolios.[4] We ask ourselves at what fair price

[3]Since there is no price for demographic risk, expectations of functionals of the intensity - such as the survival probability - under the historical measure $\mathbb{P}$ and the risk-neutral one/ones $\mathbb{Q}$ coincide. Extensions to constant risk premiums are trivial.

[4]If the reinsurer has no death contract on the same generation available for hedging, which is possible for pensioneers, he may use death contracts on - say - younger generations. The natural hedging tecnique we describe in Luciano et al. (2012b) can be used in order to determine the fair price in that case. In this paper we assume that both life and death contracts contracts on the same generation are available to the reinsurer, in order to separate our main focus, VaR and efficiency, from the availability of natural hedging strategies for reinsurers.

the reinsurer can absorb the demographic risk of the annuity. To this end, we assume that coverage of risk is done by the reinsurer up to first-order changes. It Delta-covers risk[5] by using a position in $\mathbb{N}$ death contracts on individuals of the same generation, gender and age, as in Luciano et al. (2012b). At time t, a death contract which covers the period (t, T) is priced

$$V_i^D(t,T) = \sum_{u=t+1}^{T} B(t,u)\left[S(t,u-1) - S(t,u)\right],$$

and has the following Delta:

$$\Delta_D^M(t,T) = \sum_{u=t+1}^{T} B(t,u)\left[-S_i(t,u-1)X_i(t,u-1) + S_i(t,u)X_i(t,u)\right] > 0. \quad (7)$$

The position $\mathbb{N}$ is determined so that the Delta of the portfolio made by the annuity and the death contract, $(-\Delta_A^M + \mathbb{N}\Delta_D^M)\Delta I_i(t)$, is zero:[6]

$$\mathbb{N}(t,T) = -\frac{\sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) X_i(t,u)}{\sum_{u=t+1}^{T} B(t,u)\left[-S_i(t,u-1)X_i(t,u-1) + S_i(t,u)X_i(t,u)\right]} < 0.$$

The fair cost of such coverage is the value of the death contracts needed for hedging:

$$V(t,T) = -\mathbb{N} \sum_{u=t+1}^{T} B(t,u)\left[S_i(t,u-1) - S_i(t,u)\right].$$

From now on, the transfer price C will be not smaller than V. At that price, the fund may decide to transfer a part η of its longevity risk to a reinsurer, by paying a price ηC, $\eta \in [0,1]$. If it does so, it remains exposed to the part of demographic risk which it did not transfer. Approximating the exposures to the first-order, the longevity risk of the fund is

$$(1-\eta)\Delta I_i(t) \sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) X_i(t,u).$$

In section 4 the discrepancy between C and V, i.e. the profit of the reinsurer - or the lack of competition in the reinsurance market - will play a key role in determining the efficiency of given hedging strategies.

[5]We maintain the assumption of Delta - as opposite to Delta-Gamma - coverage for all risks below. In principle, going from delta to Delta-Gamma coverage just requires the use of additional death contracts and the introduction of more equations. No major conceptual difference is at stake. For this reason, we disregard the extension in the whole paper.

[6]The reinsurer is short the death contract, since the annuity value increases when longevity is greater than forecasted, while the death value decreases. As a consequence, the increase in the payments to annuitants due to an unexpected shock in longevity is compensated by the decrease in the expected payments due to life-insurance policyholders.

3.2 Financial risk measurement

Any bond on the asset side is subject to financial risk. If for simplicity we consider zero-coupon bonds only, their sensitivity to changes in K - ΔK - is well known:

$$\Delta_B^F(t,T) = -B(t,T)\bar{X}(t,T) < 0, \tag{8}$$

$$\Gamma_B^F(t,T) = B(t,T)\bar{X}^2(t,T) > 0. \tag{9}$$

Bond values are decreasing and convex in discrepancies between the actual and forecasted interest rates.

The annuity value, which enters the liabilities, is subject to financial risk, since it is fairly priced. The effect of a change in K on the annuity value - approximated at second order - is:

$$\Delta V_i^{AF}(t) = \Delta_A^F(t)\Delta K(t) + \frac{1}{2}\Gamma_A^F(t)\Delta K^2(t),$$

where

$$\Delta_A^F(t) = -\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) < 0,$$

$$\Gamma_A^F(t) = \sum_{u=t+1}^{\omega-x} B(t,u)S(t,u)[\bar{X}(t,u)]^2 > 0.$$

The annuity is decreasing and convex in discrepancies between the actual and forecasted interest rates, exactly as the bonds are.

In order to evaluate the change in the whole value of assets and liabilities, for any specific realization of ΔK, we have to specify how the premium P of the annuity is used for asset purchases. We assume that a duration-matching strategy is pursued. The maturity T of the bonds is chosen so as to equal the annuity one, i.e.

$$\tau^* = \frac{\sum_{u=t+1}^{\omega-x} u \times S_i(t,u)B(t,u)}{V_i^A(t)}$$

Given that bonds are zero-coupon, the asset duration, before reinsurance is bought, is $P \times (T-t)$. So, the two match if and only if

$$T^* = t + \frac{P}{\tau^*}. \tag{10}$$

Once the bond duration is identified, the part of the premium which is not used for demographic-risk transfer, $P - \eta C$, is invested in bonds. The number of bonds bought is

$$n^* = \frac{P - \eta C}{B(t,T^*)}. \tag{11}$$

Everything else being equal, n^* is decreasing in η: the higher the level of reinsurance, the lower is cash available for bond purchasing. The financial risk incurred by the fund, as a consequence of this asset policy, can be evaluated at first order as follows.

$$\left[-\Delta_A^F(t) + \frac{P-\eta C}{B(t,T^*)}\Delta_B^F(t,T^*)\right]\Delta K(t) =$$

$$= \left[-\Delta_A^F(t) - \frac{P-\eta C}{B(t,T^*)}B(t,T^*)\bar{X}(t,T^*)\right]\Delta K(t)$$

$$= \left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - (P-\eta C)\,\bar{X}(t,T^*)\right]\Delta K(t). \qquad (12)$$

The expected financial return of the fund is

$$\mathbb{E}_t\left[-V_A^F(t+dt) + V_A^F(t) + \frac{P-\eta C}{B(t,T^*)}\left[B(t+dt,T^*) - B(t,T^*)\right]\right] \qquad (13)$$

$$\simeq \mathbb{E}_t\left[\Delta K(t)\right]\left[\Delta_A^F + \frac{(P-\eta C)}{B(t,T^*)}\Delta_B^F\right] = \qquad (14)$$

$$= \mathbb{E}_t\left[\Delta K(t)\right]\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - (P-\eta C)\,\bar{X}(t,T^*)\right]. \qquad (15)$$

Two extreme situations arise, when the demographic reinsurance policy is either $\eta = 0$ (strategy 1) or $\eta = 1$ (strategy 2):

1. $\eta = 0$: the fund does not transfer demographic risk. It has financial risk from assets, since $n^* = P/B$, as well as from liabilities. Financial risks are

$$\Delta I_i(t) \sum_{u=t+t+1}^{\omega-x} B(t,u)S_i(t,u)X_i(t,u),$$

$$\Delta K(t)\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - P\bar{X}(t,T)\right]. \qquad (16)$$

while expected returns equal

$$\mathbb{E}_t\left[-V_A^F(t+dt) + V_A^F(t) + \frac{P}{B(t,T^*)}\left[B(t+dt,T^*) - B(t,T^*)\right]\right] \qquad (17)$$

$$\simeq \mathbb{E}_t\left[\Delta K(t)\right]\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - P\bar{X}(t,T)\right] \qquad (18)$$

2. $\eta = 1$: the fund transferred all demographic risk. It has financial risk from assets, since $n^* = (P-C)/B$, and liabilities. This risk is equal to

$$\Delta K(t)\left[\sum_{u=t+1}^{\omega-x} B(t,u)S_i(t,u)\bar{X}(t,u) - (P-C)\,\bar{X}(t,T^*)\right]. \qquad (19)$$

The expected returns of this strategy equal

$$
\begin{aligned}
& \mathbb{E}_t \left[-V_A^F(t+dt) + V_A^F(t) + \frac{P-C}{B(t,T^*)} \left[B(t+dt,T^*) - B(t,T^*) \right] \right] \\
\simeq \quad & \mathbb{E}_t \left[\Delta K(t) \right] \left[\sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) - (P-C) \, \bar{X}(t,T^*) \right].
\end{aligned}
$$

3.3 Overall impact

Let us introduce the following notation:

$$
\begin{aligned}
\alpha \quad & : \quad = \sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) X_i(t,u) > 0, \\
\beta \quad & : \quad = \sum_{u=t+1}^{\omega-x} B(t,u) S_i(t,u) \bar{X}(t,u) > 0, \\
\nu \quad & : \quad = \beta - (P - \eta C) \bar{X}(t,T^*), \\
\gamma \quad & : \quad = \beta - P\bar{X}(t,T^*) < \beta, \\
\delta \quad & : \quad = \gamma + C\bar{X}(t,T^*) > \gamma.
\end{aligned}
$$

where α is the Delta of the portfolio with respect to mortality risk in strategy 1, while γ and δ are the Deltas of the portfolios for the two strategies with respect to financial risk. Correspondingly, ν is the financial risk of each intermediate strategy, obtained by setting $\eta \in (0,1)$.

Let C^* be the cost associated with any fixed reinsurance policy: $C^* = \eta C$.

With this notation, strategies 1 and 2 are as described in Table 1. Expected financial returns are evaluated at the end of the interval Δt and are net of the costs $C^*_{\Delta t}$ of demographic-risk transfer, obtained as $\Delta t C^*/(\omega - x)$. We denote them with μ.[7]

Strategy	n^*	C^*	**Dem risk**	**Fin risk**	**Net expected return**
1	P/B	0	$\alpha \Delta I$	$\gamma \Delta K$	$\gamma \mathbb{E}[\Delta K]$
2	(P-C)/B	C	0	$\delta \Delta K$	$\delta \mathbb{E}[\Delta K] - C^*_{\Delta t}$

Table 1: Risks and expected return

The column devoted to demographic risk obviously says that it has an higher impact under the first than under the second strategy, where it is null. The column devoted to financial risk says that bonds partially offset the effect of forecast errors in rates ΔK on liabilities. For instance, with $\Delta K < 0$, assets

[7] Notice that we could subtract the whole cost of reinsurance – which lasts for the whole annuity maturity, $\omega - x$ – to compute financial returns. This would lower financial returns. The model can accommodate any splitting of the reinsurance cost over the maturity of the annuity.

increase in value when γ and (a fortiori) δ are positive. Since $\delta > \gamma$, the offsetting effect is larger when the whole amount of the premium P is used to buy bonds, under strategy 1, than under strategy 2.

4 VaR

In order to go from the impact of a specific forecast error in interest rates or mortality to an overall risk evaluation, which takes the distribution of the forecast errors into account, we compute VaR. With that, we can study the risk-return trade-offs of the two strategies – and all the intermediate ones. We aim at going from the change in the portfolio net exposure corresponding to a specific difference between forecasted and actual mortality ΔI_i or interest rate ΔK to a synthetic risk-and-return couple valid for every scenario $(\Delta I_i,\ \Delta K)$. In order to reconstruct a risk/return tradeoff, without losing the information about the effect of the two sources of risk, in this section we proceed in three steps. We first recognize the link between the scenario-based risk representation and a VaR risk-measurement for each risk factor. Then, we pass from the VaR of the factor to the VaR of the portfolio strategy. Third, we sum up the VaRs due to financial and demographic risk to obtain the Overall VaR.

4.1 One-standard deviation shocks and VaRs

This section formalizes the move from risk-factor changes to risk appraisal through VaR. The main advantage of the Delta approach taken here consists in making the factor approach to VaR computation with Gaussian innovations possible. To this end, observe first that the expected values of the risk-factors changes, $\Delta I_i = \Delta I_i(t + \Delta t)$ and $\Delta K = \Delta K(t + \Delta t)$, are equal to the expected values of the mortality intensity and interest rate, $\lambda_i(t + \Delta t)$ and $r(t + \Delta t)$, which we computed above, in (1) and (4), net of the corresponding forward rate. The variances are the ones computed in (2) and (5). So, using (1), (4), (2) and (5), we can compute $\mathbb{E}[\Delta I_i], \mathbb{V}ar[\Delta I_i], \mathbb{E}[\Delta K], \mathbb{V}ar[\Delta K]$.

Consider a positive or negative one-standard-deviation shock on the longevity of generation i and on interest rates:

$$\Delta I_i = \mathbb{E}[\Delta I_i] \pm 1 \times \sqrt{\mathbb{V}ar[\Delta I_i]}, \tag{20}$$

$$\Delta K = \mathbb{E}[\Delta K] \pm 1 \times \sqrt{\mathbb{V}ar[\Delta K]}. \tag{21}$$

Since both the intensity and the interest rate are Gaussian, looking at a one-standard-deviation shock means to examine the worst occurrence for I and K in 84% or 16% of the cases. Expressions (20) and (21) give the *VaR of the risk factors* at the level of confidence 84% - if we take $-1 \times \sqrt{\mathbb{V}ar[\Delta I_i]}$ - and 16%, if we take $+1 \times \sqrt{\mathbb{V}ar[\Delta I_i]}$. In general, we can fix a confidence level $1 - \epsilon$ (say 99%, 95%, 84%) or ϵ (1%,5%, 16%) at which the VaR of the risk factors can be evaluated, by choosing appropriately the constant in front of the standard deviation. Let $n(\epsilon)$ be that constant. The VaR of the two risk factors at the confidence level $1-\epsilon$ is

$$VaR_{1-\epsilon}(\Delta I_i) = \mathbb{E}[\Delta I_i] - n(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]}, \tag{22}$$

$$VaR_{1-\epsilon}(\Delta K) = \mathbb{E}[\Delta K] - n(\epsilon)\sqrt{\mathbb{V}ar[\Delta K]}. \tag{23}$$

However, in the end we are interested in the *VaR of the portfolio*, not in the VaR of the risk factors. According to Table 1, the realizations of the portfolio gains/losses are of the type $k\Delta I_i$ or $k\Delta K$, where the constant k can be either positive or negative ($k = \alpha, \beta, \gamma, \delta$). An increase in the risk factor corresponds to a portfolio loss if $k < 0$, to a gain if $k > 0$. Hence, we consider as "worst case scenarios" the outcomes in the left tail of the distribution of ΔI_i and ΔK – and thus $VaR_{1-\epsilon}(\cdot)$ – when $k > 0$ and the outcomes in the right tail – thus $VaR_\epsilon(\cdot)$ – when $k < 0$. With a slight abuse of terminology, let us define $VaR_M(\cdot)$ the difference between the demographic VaR-component and its expected value:

$$VaR_M(k;\epsilon) = \begin{cases} kVaR_{1-\epsilon}(\Delta I_i) - k\mathbb{E}[\Delta I_i] = -kn(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]} & \text{if } k > 0, \\ kVaR_\epsilon(\Delta I_i) - k\mathbb{E}[\Delta I_i] = +kn(\epsilon)\sqrt{\mathbb{V}ar[\Delta I_i]} & \text{if } k < 0. \end{cases}$$

Similarly for the financial VaR-component:

$$VaR_F(k;\epsilon) = \begin{cases} kVaR_{1-\epsilon}(\Delta K) - k\mathbb{E}[\Delta K] = -kn(\epsilon)\sqrt{\mathbb{V}ar[\Delta K]} & \text{if } k > 0, \\ kVaR_\epsilon(\Delta K) - k\mathbb{E}[\Delta K] = kn(\epsilon)\sqrt{\mathbb{V}ar[\Delta K]} & \text{if } k < 0. \end{cases}$$

Table 2 reports the values of the financial and demographic VaR-component for each strategy.

Table 2: Demographic and Financial VaR-components for the two strategies

	Contribution to the strategy VaR	
Strategy	Demographic VaR-component	Financial VaR-component
1	$VaR_M(\alpha;\epsilon) + \alpha\mathbb{E}[\Delta I_i]$	$VaR_F(\gamma;\epsilon) + \gamma\mathbb{E}[\Delta K]$
2	0	$VaR_F(\delta;\epsilon) + \delta\mathbb{E}[\Delta K]$

If we aggregate the appropriate scenario-based risks or VaRs (where appropriate stands for "which use $VaR_\epsilon(\cdot)$ or $VaR_{1-\epsilon}(\cdot)$, as needed") taking into account the diversification benefit due to our independence assumption, we obtain the *Overall VaR* ($OVaR$):

$$OVaR(k_M; k_F; \epsilon) = k_M\mathbb{E}[\Delta I_i] + k_F\mathbb{E}[\Delta K] - \sqrt{(VaR_M(k_M))^2 + (VaR_F(k_F))^2}, \tag{24}$$

where k_M is α for strategy 1 and 0 for strategy 2, k_F is γ for strategy 1 and δ for strategy 2. $VaR_M(\cdot)$ and $VaR_F(\cdot)$ are evaluated at the same confidence level ϵ. Formula (24) represents the worst case outcome for the change in the value of the net position of the fund. This is why we take the negative sign in front of the square root, because bad outcomes are those associated to negative changes of the net exposure. In what follows, for simplicity, we will always focus on the absolute value of $OVaR$ itself: greater values will then mean greater risks. We report Overall VaR for the competing strategies in Table 3, together with the corresponding financial expected return. This representation opens the way to representing the trade-offs of the strategies in a familiar way, by associating to each strategy a point in the plane (Overall VaR, Expected Financial Return).

Table 3: Overall VaR and Expected Financial Return for strategies 1 and 2

Strategy	**($OVaR$, μ) combination**
1	$\left(\left\vert\alpha\mathbb{E}[\Delta I_i]+\gamma\mathbb{E}[\Delta K]-\sqrt{(VaR_M(\alpha))^2+(VaR_F(\gamma))^2}\right\vert,\gamma\mathbb{E}[\Delta K]\right)$
2	$(\vert\delta\mathbb{E}[\Delta K]+VaR_F(\delta)\vert,\delta\mathbb{E}\left[\Delta K\right]-C_{\Delta t})$

5 Risk-return frontier; efficiency versus inefficiency

We represent the limit strategies 1 and 2, as well as the intermediate ones – in which demographic risk is partially reinsured – in the plane (Overall VaR, Expected Financial Return). Expected financial returns, net of the cost of reinsurance are:

$$\mu=\nu\mathbb{E}[\Delta K]-C^*_{\Delta t}.$$

Strategies are characterized by the following couple of values, when η goes from 1 to 0:

$$\left(\left|k_M\mathbb{E}[\Delta I_i]+k_F\mathbb{E}[\Delta K]-\sqrt{(VaR_M((1-\eta)\alpha))^2+(VaR_F(\nu))^2}\right|,\nu\mathbb{E}[\Delta K]-C^*_{\Delta t}\right). \quad (25)$$

For any given confidence level ϵ for $OVaR$, this is a curve between point P_2, which represents strategy 2, and point P_1 for strategy 1 (Figure 1).

[Insert Here Figure 1]

Notice that the derivatives or Deltas of the portfolio change with respect to demographic and longevity risk (with their signs) are $(1-\eta)\alpha$ and ν, which we denote as Δ^M and Δ^F. Since $\Delta^M\geq 0$ the effect of demographic risk Δ^M is null at P_2, where $\eta=1$, and positive at all other points of the line, where $\eta<1$. This means that, when η decreases from 1 to 0 and we move from strategy 2 towards 1, demographic risk increases. The financial Delta Δ^F is positive between P_2 and the point Q, it is negative between Q and P_1. The point Q where the Delta of the portfolio with respect to financial risk is null is characterized by the reinsurance level η which solves the equation $\Delta^F(\eta)=0$. Its value is

$$\bar{\eta}=\frac{\Delta^F_A+P\bar{X}(t,T^*)}{C\bar{X}(t,T^*)}.$$

We concentrate on the most interesting case in which $0 < \bar{\eta} < 1$.[8] Moving from P_1 to P_2 the demographic component of $OVaR$ decreases, since Δ^M does. The financial risk component instead decreases first, since between P_1 and Q its absolute value decreases to reach zero in Q, and then increases, since between Q and P_2 Δ^F is always greater than zero and increasing. In order to analyze the effects of this move along the frontier, we denote with H the point of the frontier itself where the Overall VaR reaches its minimum. This point may be interior or may coincide with P_2. This depends on the value of η at H, η^*. If η^* does not lie inside the interval [0,1), H coincides with P_2. In this case, the line between P_2 and P_1 is always positively sloped. This is the situation depicted in Figure 1. It means that increasing the reinsurance level leads to both lower risk and lower returns, as intuition would command. As a whole, the hedging strategy is efficient. If η^* lies inside the interval [0,1), H is interior. In this case the part of the frontier between H and P_2 is negatively sloped. Reducing η, i.e. reinsuring less demographic risk, always increases the expected portfolio return. This is the situation depicted in figure 2. The peculiarity of the frontier, which defies our naive intuition about the usefulness of hedging longevity risk by transferring it, emerges exactly in this case.

When H and P_2 coincide - as in Figure 1 - VaR decreases with η, and the strategies are efficient along the whole frontier, both between Q and P_2 and between Q and P_1, even though there $\Delta^F > 0$. When H is interior - as represented in Figure 2 - the reduction in demographic risk counterbalances the increase in financial risk only between Q and H.

[Insert here Figure 2]

Overall risk decreases, while returns go down. Between H and P_2, instead, the increase in financial risk overcomes the reduction in demographic one. The overall effect is an increase of portfolio VaR which makes the whole set of strategies in this part of the curve inefficient.

Briefly, when moving from Q to P_2 the VaR component due to demographic risk decreases, while the one due to financial risk increases. Where the first effect prevails, the frontier is positively sloped and the transfer is *efficient*. Where the second does – i.e. between H and P_2 – the frontier is negatively sloped and the transfer is *inefficient*. In the latter case, each point on the curve between P_2 and H represents a strategy which is dominated by the corresponding strategy (same $OVaR$) on the upper part of the frontier, since the latter has higher return. An example of such a situation is in Figure 2, where strategy p is dominated by p'. This inefficiency cannot be captured by those approaches which do not take a holistic view of risk. The possible existence of an *inefficient* part of the frontier,

[8] If $\bar{\eta} \geq 1$, then Δ^F is always negative. In this case, the financial risk component of $OVaR$ is always decreasing for $0 < \eta < 1$, as the demographic one is: the frontier is always efficient. If $\bar{\eta} \leq 0$, Δ^F is always positive. In this case, the same reasoning of the case in which $0 < \bar{\eta} < 1$ applies, since $OVaR$ can increase or decrease with $0 < \eta \leq 1$. The frontier may present an efficient and an inefficient part. Indeed, strategy $\eta = 0$ might also constitute the only efficient strategy, if it coincides with the minimum value of $OVaR$ among all possible strategies.

made by dominated strategies, depends not only on the coefficients in Table 1, but also on the characteristics of the risk factors distributions, on the VaR confidence level and on the cost of reinsurance.[9] The condition for the existence of an inefficient part of the frontier is

$$\begin{aligned} &\exists \arg\min_{\eta} OVaR(k_M; k_F; \epsilon) \neq 1 \\ &\text{s.t. } 0 \leq \eta \leq 1. \end{aligned}$$

which in turn depends on whether the derivative of the absolute value of $OVaR$ with respect to η is neutralized at $0 \leq \eta^* < 1$ or not. This derivative takes the value:

$$\begin{cases} (1-\alpha)\mathbb{E}[\Delta I_i] + \mathbb{E}[\Delta K]C\bar{X}(t,T^*)+ \\ -\frac{2(1-\alpha)^2\eta Var(\Delta I_i)(n(\epsilon))^2 + 2\nu C\bar{X}(t,T^*)(n(\epsilon))^2 Var(\Delta K)}{2\sqrt{((VaR_M(k_M))^2+(VaR_F(k_F))^2}} & \text{if Overall VaR } \geq 0 \\ -(1-\alpha)\mathbb{E}[\Delta I_i] - \mathbb{E}[\Delta K]C\bar{X}(t,T^*)+ \\ +\frac{2(1-\alpha)^2\eta Var(\Delta I_i)(n(\epsilon))^2 + 2\nu C\bar{X}(t,T^*)(n(\epsilon))^2 Var(\Delta K)}{2\sqrt{((VaR_M(k_M))^2+(VaR_F(k_F))^2}} & \text{if Overall VaR } < 0. \end{cases}$$

Neutralizing it, we obtain the value(s) at which local minima lie. The corresponding equation is highly non linear and must be solved numerically. If we find that none of the solutions lies between 0 and 1, then the whole frontier is efficient. Otherwise, we have an inefficient part.

5.1 Efficiency and optimality

As an example of the application of the efficiency just pointed out, let us now introduce a decision criterion for optimal hedging of the insurance portfolio, which works on the efficient part of the frontier. We define the risk-return preferences of the fund through a utility function defined on the plane (Overall VaR, Expected Return). This choice does not pretend to be axiomatically based, but simply to be consistent with a VaR-based measurement of risk. Given these preferences, we can choose $\eta^* \in [0,1]$ which maximizes an expected utility of the type:

$$U(\mu, OVaR(k_M; k_F; \epsilon), \xi), \tag{26}$$

with $U' > 0, U'' < 0$, where ξ is a parameter (or, possibly, a set of parameters) describing the risk attitude of the fund. Graphically, the best strategy is identified as the point on the efficient part of the frontier that crosses the highest possible indifference curve, as represented in Figure 3. This point determines the optimal level of reinsurance demanded by the fund.

[Insert Here Figure 3]

[9] The reason why there is still financial risk left, in spite of duration matching, is that the duration itself is a classical one, not the Delta or Delta-Gamma duration matching which can be performed in the Hull-White setting (see for instance Avellaneda (2000)). The latter one would eliminate any riskiness up to first or second order approximations, but would leave no room for exploiting the risk-return trade-off, i.e. for optimization.

5.2 Larger portfolios

Up to now we have limited ourselves to a simple portfolio, made by one annuity on the liability side, bonds and cash on the asset side. A portfolio in which several life contracts are sold on the same generation can be easily described, since it would depend on the same demographic risk factor K and the same interest rate factor I. Only the Greeks should be adjusted so as to reflect the presence of more than one contract. Abstracting from reinsurance considerations, in the presence of n annuities and m death contracts on the same generation, for instance,[10] we would have the following first-order value change due to the mortality risk factor:

$$(n\Delta_A^M + m\Delta_D^M)\Delta I_i(t),$$

where the death contract Greek has been defined in (7). In case the same generation had both life and death contracts in force with the insurer, the Overall VaR due to the generation would then be easy to compute too, according to formula (24) with $k_M = n\Delta_A^M + m\Delta_D^M$.

In the presence of several generations, our estimates can be easily extended if the factors affecting the mortality of several generations are perfectly correlated and all independent from the financial risk factor. If the correlation between the intensities of different generations is not one, the above formulas represent however an upper bound for the VaR of the insurer's portfolio.

In all cases, the efficiency problem remains. In order to appreciate the VaR efficiency and its effect on the optimal reinsurance policies - as representative of hedging policies in general - let us now introduce and comment an example calibrated on UK mortality and financial data.

6 VaR, efficient and inefficient frontier on UK data

Let us compute the VaR and study efficient versus inefficient strategies using data from the UK market. To be specific, we consider a whole-life annuity sold on a UK male aged 65 at strategy inception, December 30, 2010; we take financial data from the UK Government market on the same date. We presume that the revenues from annuity sales are invested in UK Government bonds whose maturity matches the annuity duration. The Hull-White model is calibrated to zero-coupon bond prices at the same date. Under the risk-neutral measure its parameters are $g = 6.32\%$, $\theta = 16.33\%$, $\Sigma = 3.32\%$, while $r(0) = 0.42\%$. The market price of risk is chosen so that the long-run mean under the historical measure is around 4%, which is the average UK short (1-month) rate in the previous 10 years. The survival rates are calibrated from projected IML92 tables. [11] For

[10] See also Luciano et al. (2012b).

[11] IML92 projected rates are derived from an underlying model which differs from ours. Our choice to fit our mortality model to these rates is driven by the idea that our framework can also describe with a limited number of parameters survival curves obtained with complex and possibly accurate projection methods.

the generation we consider, the model parameters are $a_i = 10.94\%, \sigma_i = 0.07\%$ and $\lambda_i(0) = 0.885\%$. Table 4 summarizes all relevant parameters. Table 5

Table 4: Calibrated parameters

Symbol	Value
Financial risk	
g	6.32%
Σ	3.32%
θ	16.33%
r(0)	0.42%
Demographic risk	
a_i	10.94%
σ_i	0.07%
$\lambda_i(0)$	0.885%

reports prices and Deltas of the instruments we use in the example.

Table 5: Risk exposures and prices of instruments

Figure	Symbol	Value
Annuity		
Price	V^A	13.14
Exposure to longevity risk	Δ_A^M	-378.72
Exposure to financial risk	Δ_A^F	-85.03
10-year bond		
Price	B(0,9.69)	0.725
Exposure to financial risk	Δ_B^F	-5.25

The fair price of the annuity − which is also its selling price − is $V^A = P = 13.14$.[12] Being short the annuity, which has exposures $\Delta_A^M = -378.72$ and $\Delta_A^F = -85.03$ the fund remains exposed to both risk factors change. The fund operates on the financial market using a bond whose maturity is computed according to (10) and is $T^* = 9.69$. We assume the existence of such a bond, which is priced $B(0, 9.69) = 0.725$ and has $\Delta_B^F = -5.25$. We evaluate the hedging strategies we described above at an horizon $\Delta t = 1$ year. The longevity risk factor $I(1)$ is expected to be positive, $\mathbb{E}_0[I(1)] = 2.73 * 10^{-7}$ while its variance is $\mathbb{V}ar_0[I(1)] = 5.47 * 10^{-7}$. Demographic risk can be transferred to a reinsurer at its fair price $C = 3.61$. The expected value of the financial risk factor under the historical measure is slightly negative, equal to $\mathbb{E}_0[K(1)] = -0.10\%$, while its variance is $\mathbb{V}ar_0[K(1)] = 0.00087$. For the sake of realism, we charge expected

[12]In the computation, we considered an extreme age $\omega = 110$ years.

returns not with the whole reinsurance cost, but with the part which refers to the cover of the horizon considered. As a consequence, financial returns are $\mathbb{E}_0[K(1)] - C_{\Delta t}$ and $C_{\Delta t} = 0.0803$.

The coefficient δ is positive, 16.05, while γ is negative, -10.10. In Table 6 we report the exposures, the expected financial return net of the reinsurance cost and the remaining liquidity of strategies 1 and 2. Strategy 1 invests all P in

Table 6: Risk exposures, VaR and expected returns

Figure		Strategy	
	Symbol	1	2
Number of bonds	n^*	18.12	13.14
Cost of reinsurance	C^*	0	3.61
Exposure to longevity risk	$\alpha/0$	378.72	0
Exposure to financial risk	γ/δ	-10.10	16.05
Expected financial return	μ	0.010	-0.096
Demographic VaR-component 99.9%	$VaR_M(\alpha; 99.9\%) + \alpha\mathbb{E}[\Delta I_i]/0$	0.84	0
Financial VaR-component 99.9%	$VaR_F(\gamma; 99.9\%) + \gamma\mathbb{E}[\Delta K]/$ $VaR_F(\delta; 99.9\%) + \delta\mathbb{E}[\Delta K]$	0.88	1.44
Overall $VaR_{99.9\%}$	$OVaR_{99.9\%}$	1.22	1.44

$n^* = 18.12$ bonds. It offers a positive expected return, 0.01, since the fund has negative exposure to financial risk ($\alpha = -10.10$) and the expected value of the risk factor is negative too. The overall VaR, computed at a one-year horizon and at a 99.9% confidence level, is 1.22. The financial VaR-component (0.88) is slightly more prominent than the demographic one (0.84). The presence of a diversification benefit is evident, since $OVaR$ is way lower than the sum of the VaR-components (1.72).
Strategy 2 hedges against longevity risk and invests the remaining resources $P - C$ to buy $n^* = 13.14$ bonds. Comparing the two strategies, we find a first interesting result. As expected, μ is lower in strategy 2 (-0.096 vs. 0.01), partly due to the cost of reinsurance which is paid for the longevity risk transfer. However, despite reinsurance against demographic risk, overall VaR increases from 1.22 of strategy 1 to 1.44 of strategy 2. This happens because reinsuring against demographic risk prevents the fund from offsetting the financial risk due to the annuity position by purchasing enough bonds. Financial risk is indeed the only source of risk in strategy 2.
In the end, strategy 1 dominates strategy 2, showing higher expected returns and lower risk, measured through $OVaR$.

6.1 Choosing the optimal strategy

Let us now turn to the analysis of strategy selection when all the intermediate proportional reinsurance strategies with $\eta \in (0,1)$ can be pursued. Figure 4 represents the set of all possible strategies in the plane (Overall VaR, Expected Financial Return).

[Insert Here Figure 4]

It is a curve between P_1, which represents strategy 1 and P_2, which shows the risk/return couple of strategy 2. As η increases (moving from P_1 towards P_2), demographic risk exposure decreases (and so does the demographic VaR-component), to reach 0 at P_2. The financial VaR-component decreases in the first part of the curve, and reaches its minimum (which is zero) at point Q. After Q, it starts increasing until P_2. If we specify a utility function for the fund, defined with respect to expected returns and overall VaR, we know that the fund can optimally choose between the competing strategies the strategy that maximizes utility. Let us consider for example a simple expected utility function

$$U\left(\mu, OVaR(k_M; k_F; 99.9\%)\right) = \mu - \xi \left(OVaR(\cdot;\cdot;99.9\%)\right)^2,$$

in which $\xi > 0$ is a measure of risk aversion correlated with the risk aversion coefficient. Let us set $\xi = 0.05$. The dotted line in Figure 4 represents the highest indifference curve that crosses the set of admissible strategies. The tangency point O between the two curves determines the optimal fund strategy according to this utility criterion. This optimal strategy consists in reinsuring $\eta_O = 27.91\%$ of the longevity exposure (at a total cost of 1.00, of which 0.022 imputed to the first year of the contract) and buying 16.73 bonds. It implies more exposure to demographic (0.61) than to financial (0.25) risk and it is characterized by $U_O = -0.0409$ - which is higher than the utility of strategy 1 - $OVaR = 0.65$, and an instantaneous expected return $\mu = -0.02$.

6.2 Efficient and inefficient outcomes

In our UK-calibrated case transferring longevity risk may increase $OVaR$. With the parameters at hand, even when the price of the transfer is fair, part of the frontier is inefficient.[13] Figure 4 clearly shows this feature.

The curve connecting strategies 1 and 2 is positively sloped in its upper part. Starting from strategy 1 and increasing the reinsurance level, $OVaR$ first goes down, since both its components do, up to point Q. $OVaR$ reaches its minimum at H (0.49). Between Q and H the decrease in the demographic VaR-component offsets the increase in the financial one. Point H represents the strategy in which 46% of the demographic risk is transferred. Beyond that

[13] If we consider a loading factor on the cost of reinsurance, η^* lowers and the set of inefficient strategies enlarges.

point, the curve becomes negatively sloped and $OVaR$ starts increasing while expected return continues to lower. The financial VaR-component increases so much that it offsets the relief in demographic risk. All the combinations of risk and return on the part of the frontier between H and P_2 are clearly suboptimal, since for each of them a strategy with same $OVaR$ and higher expected financial return exists. We conclude that any transfer in excess of 46% is inefficient. We could not have captured this effect by looking only at financial or longevity risk, or modelling them separately and differently.

7 Conclusions

This paper explores analytically the risk-return trade-off of a pension fund, when risk is measured by VaR. By so doing, it separates efficient from inefficient hedging strategies. It takes a holistic view of financial and longevity risk management, since demographic risk transfer impacts on interest-rate risk exposure. We build a (VaR, Expected Return) frontier, where VaR comes from both financial and longevity shocks. Our main result is that transfer of longevity risk may decrease or increase VaR in absolute value, since it decreases its longevity part, but may either increase or decrease the financial one. This phenomenon cannot be captured by those approaches which do not take a holistic view of risk. We provide a fully calibrated example - which reproduces a 65-year old UK annuity coverage - which shows that if demographic risk can be transferred at a fair value, any transfer in excess of around 46% is inefficient. This happens because the risk/return frontier includes strategies which are inefficient from the point of view of fund managers.

Our conclusions provides a rationale for some of the recent mortality-transfer deals, which cover only part of the mortality risk of the underlying portfolio. Obviously, we take a stylized view of the problem. We cover a single annuity, which stands for a homogeneous group of them. On the asset side we allow only for bond purchasing. Reinsurance through derivatives is not formalized. Last, we concentrate on a single generation (as Delong et al. (2008) and Cox et al. (2013) do) and disregard minimum capital requirements. All the realistic features, such as a richer liabilities portfolio with idiosyncratic risk or a richer investment opportunity set, or more complex liability-risk transfer, using q-forwards or s-forwards, are left for future extensions. Multiple-generation versions are an obvious extension too.

Even in this stylized setting, we feel that the VaR frontier convey an important policy message. There is general consensus on the fact that longevity risk, as well as other risks, is "too large to be managed by one sector of the society (IMF, 2012)" and that there should be better risk sharing between the private business sector, the public sector and individuals. Our set up shows that, even for a single category of agents, namely the private business sector, not all risk sharing strategies are optimal, especially in the presence of illiquid markets for risk transfer, in which profits may be high. This is one more reason for fostering the development of alternative risk transfer possibilities, which may lower the transfer price.

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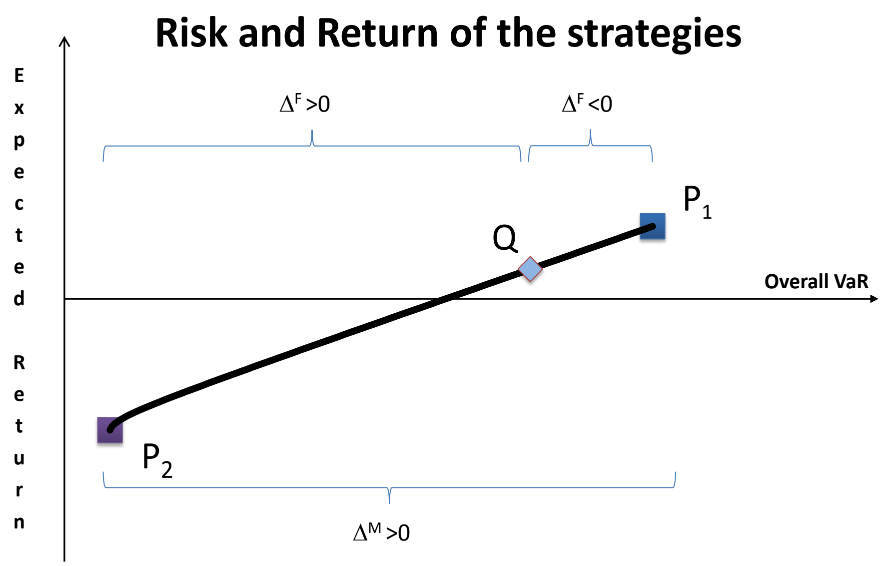

Figure 1: This figure shows the risk-return combinations of strategies 1 and 2 and intermediate ones, for all the possible values of η. The strategies are represented by the black curve. H in this case coincides with P_2 and hence the whole frontier is efficient.

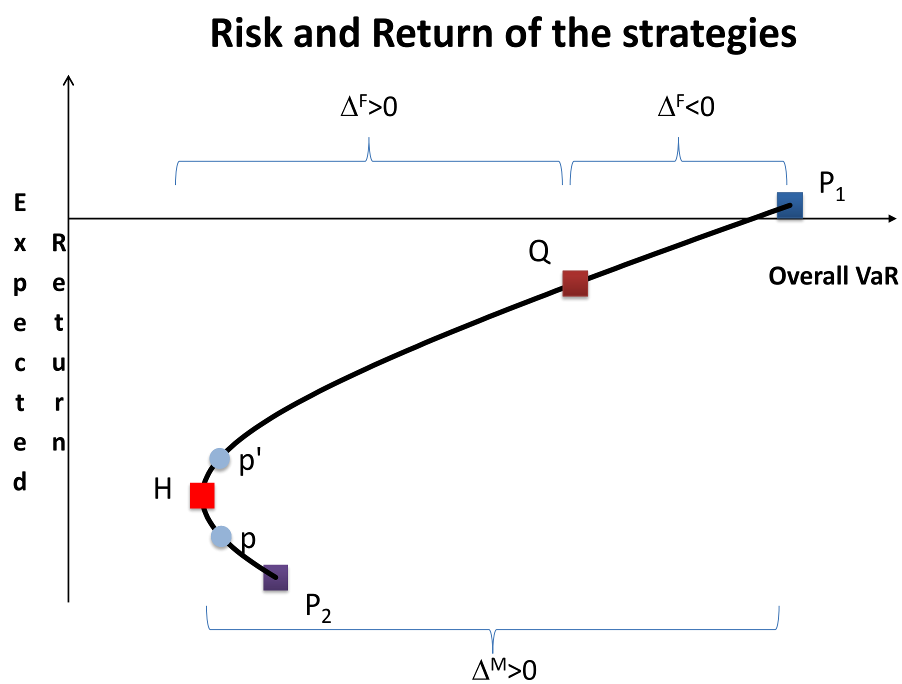

Figure 2: This figure shows the risk-return combinations of the set of strategies 1 and 2 and the intermediate strategies for all the possible values of η. The frontier has an inefficient part, which comprises the strategies depicted between H and P_2.

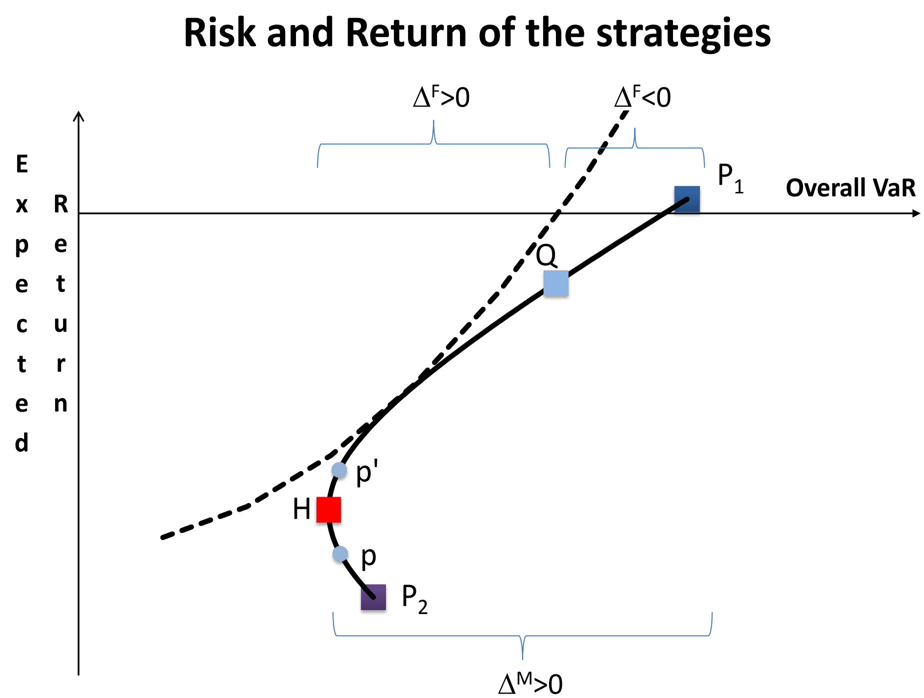

Figure 3: This figure shows the risk-return combinations of strategies 1 and 2 and the intermediate ones for all the possible values of η. On the horizontal axis the $OVaR$ at a certain level ϵ is reported, while the expected financial return net of reinsurance costs lies on the vertical axis. The strategies are represented by the black solid line. The dotted curve is the highest indifference curve which is tangent to the set of strategies. The optimal strategy lies at the intersection between the curve and the line.

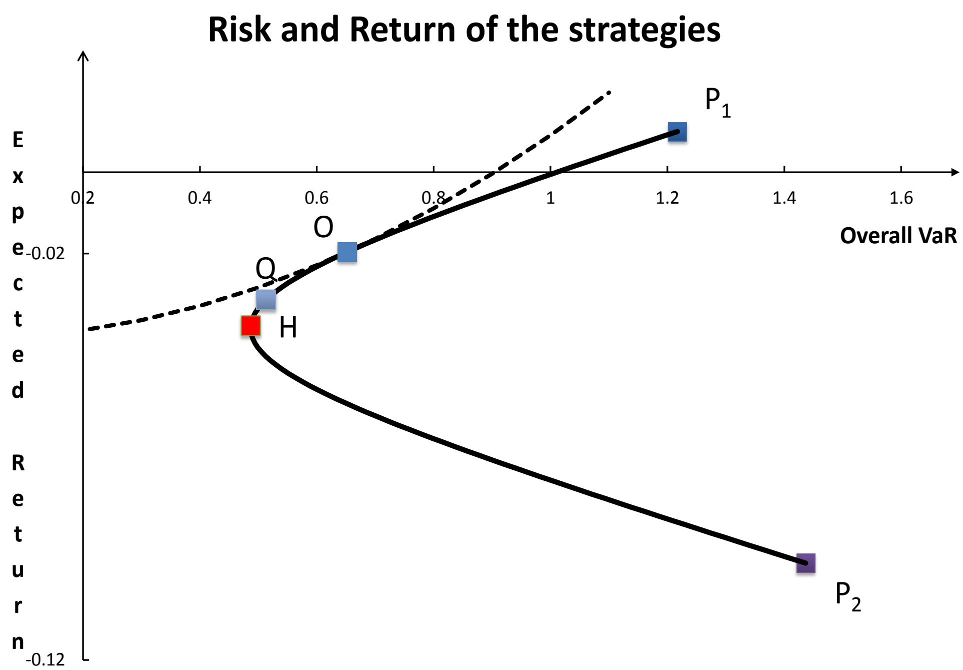

Figure 4: This figure shows the risk-return combinations of the set of strategies. On the horizontal axis the $OVaR$ at level 99.9% is reported, while the expected financial return net of reinsurance costs lies on the vertical axis. The dotted line represents the highest possible indifference curve of the utility function that crosses the set of strategies.