Stability of Hybrid Model Predictive Control

M. Lazar, W.P.M.H. Heemels, S. Weiland, A. Bemporad

Abstract

In this paper we investigate the stability of hybrid systems in closed-loop with Model Predictive Controllers (MPC) and we derive *a priori* sufficient conditions for Lyapunov asymptotic stability and exponential stability. A general theory is presented which proves that Lyapunov stability is achieved for both *terminal cost and constraint set* and *terminal equality constraint* hybrid MPC, even though the considered Lyapunov function and the system dynamics may be discontinuous. For particular choices of MPC criteria and constrained Piecewise Affine (PWA) systems as the prediction models we develop novel algorithms for computing the terminal cost and the terminal constraint set. For a quadratic MPC cost, the stabilization conditions translate into a linear matrix inequality while, for an ∞-norm based MPC cost, they are obtained as ∞-norm inequalities. It is shown that by using ∞-norms, the terminal constraint set is automatically obtained as a polyhedron or a finite union of polyhedra by taking a sublevel set of the calculated terminal cost function. New algorithms are developed for calculating *polyhedral* or piecewise polyhedral positively invariant sets for PWA systems. In this manner, the on-line optimization problem leads to a mixed integer quadratic programming problem or to a mixed integer linear programming problem, which can be solved by standard optimization tools. Several examples illustrate the effectiveness of the developed methodology.

Index Terms

Hybrid systems, Piecewise Affine systems, Model Predictive Control, Lyapunov stability.

M. Lazar and S. Weiland are with the Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, E-mail: `m.lazar@tue.nl; s.weiland@tue.nl.`

W.P.M.H. Heemels is with the Embedded Systems Institute, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, E-mail: `maurice.heemels@esi.nl.`

A. Bemporad is with the Dipartimento di Ingegneria dell'Informazione, Universitá di Siena, Via Roma 56, 53100 Siena, Italy, E-mail: `bemporad@dii.unisi.it.`

I. INTRODUCTION

Hybrid systems provide a unified framework for modeling complex processes that include both continuous and discrete dynamics. The large variety of practical situations where hybrid systems are encountered (e.g., physical processes interacting with discrete actuators) led to an increasing interest in modeling and control of hybrid systems. Several modeling formalisms have been developed for describing hybrid systems, such as Mixed Logical Dynamical (MLD) systems [1] or Piecewise Affine (PWA) systems [2], and several control strategies have been proposed for relevant classes of hybrid systems. In particular, PWA systems have become popular due to their accessible mathematical description on one hand, and their ability to model a broad class of hybrid systems [3], [4] on the other. Many of the control schemes for hybrid systems are based on Model Predictive Control (MPC), e.g., as the ones in [1], [5–7]. MPC, also known as *receding horizon control*, is a control strategy that offers attractive solutions for industry, e.g., see [8] for a recent survey of industrial MPC controllers. Initial MPC algorithms were exclusively designed for linear systems and many ideas were soon suitably generalized to nonlinear systems [9]. As a future objective, it has been pointed out in the survey [9] that many system theoretic concepts, as well as control strategies like model predictive control, require re-examination for the class of hybrid systems. More precisely, hybrid MPC faces two difficult problems, which cannot be handled using the tools developed for linear or nonlinear models. Firstly, the computational complexity of the constrained optimization problem that has to be solved on-line and, secondly, guaranteeing closed-loop stability. In this paper we focus on the latter problem and we aim at deriving sufficient conditions that guarantee Lyapunov stability, attractivity and exponential stability for a general class of hybrid models and MPC optimization criteria. Note that many of the hybrid MPC schemes e.g., [1], [6], [7], have only been proven to guarantee attractivity, while Lyapunov stability [10–12] is a desirable property from a practical point of view as well. This is due to the fact that if attractivity alone is ensured, then in principle, an arbitrarily small perturbation from the equilibrium may cause the state of the closed-loop system to drift far away by a fixed distance before converging back to the origin.

In the literature, a hybrid MPC scheme is based on the optimization of a cost function that is defined using mainly quadratic forms, e.g. [1], [7] or $1, \infty$-norms, e.g. [5], [6]. If a quadratic cost function is used, the MPC optimization problem leads to a Mixed Integer Quadratic Programming

(MIQP) problem. An option to guarantee attractivity in this case is to impose a *terminal equality constraint* [1]. However, this method has the disadvantage that the predicted state must be brought to the origin in finite time. This requires that the PWA system is controllable, while stabilizability should be sufficient in general. Moreover, a longer prediction horizon may be needed for ensuring feasibility of the MPC optimization problem, which increases the computational complexity. Controllers with reduced complexity are proposed for this case in [13], but convergence can only be established by an *a posteriori* analysis. Although the terminal equality constraint method has been proven to guarantee attractivity [1], a proof of Lyapunov stability is missing for hybrid systems. Also, some quadratic cost hybrid MPC schemes, such as the one in [13], rely on the result of [9] (which uses continuity of the MPC value function) to claim stability. Since continuity of the value function is not guaranteed in the hybrid case, such results only guarantee attractivity in general. Sorting this aspect out precisely is one of the main topics in this paper. In the case when the 1-norm or the ∞-norm is used to define the cost function, the MPC optimization problem leads to a Mixed Integer Linear Programming (MILP) problem. An *a priori* heuristic test for guaranteeing attractivity of ∞-norm based MPC of PWA systems has been developed in [5] and an *a posteriori* stability check has been proposed in [14]. The a posteriori check is based on computing explicitly the PWA closed-loop dynamics and checking stability afterwards using the theory of [15], [16]. No indication is available how to adapt the original MPC set-up in case that the closed-loop system is unstable. The use of an a posteriori stability check emphasizes the need for conditions that guarantee stability in hybrid MPC. The inclusion of such conditions in the MPC design (i.e. a priori) would yield a major advantage. This is one of the motivations for this work.

In this paper we derive *a priori* sufficient conditions for asymptotic stability (including Lyapunov stability) of both terminal cost and constraint set and terminal equality constraint hybrid MPC. We present a general theory for a wide class of hybrid models and MPC cost functions and we show that Lyapunov stability can be achieved even though the value function and the system dynamics are discontinuous. New methods for calculating the terminal cost and the terminal constraint set are developed for the particular case of constrained PWA systems. In the case of a quadratic cost, the conditions are obtained in the Linear Matrix Inequalities (LMI) form and thus, the terminal weight(s) can be calculated using semi-definite programming. For an ∞-norm based cost, the conditions are specified using ∞-norm inequalities, which lead to a constrained optimization problem that has to be solved off-line. One of the advantages of using ∞-norms is that the terminal constraint set can be automatically obtained as a polyhedron or a finite union of polyhedra by taking a sublevel set of the calculated terminal cost function. We also develop new algorithms for calculating positively invariant sets for feedback controlled PWA systems. These algorithms provide the means to come up with *polyhedral* positively invariant sets in the case of quadratic forms based hybrid MPC and thereby obtaining an MIQP optimization problem.

The paper is organized as follows. Section II deals with preliminary definitions and Section III provides a precise problem formulation. Section IV deals with discrete-time Lyapunov stability, and the results regarding stability of hybrid MPC are given in Section V. For the case of constrained PWA systems, methods for calculating the terminal cost, the terminal constraint set and the value of the prediction horizon are developed in Section VI and Section VII for hybrid MPC based on quadratic costs and for hybrid MPC based on ∞-norms, respectively. The special case of terminal equality constraint hybrid MPC is addressed in Section VIII and the conclusions are summarized in Section IX.

II. Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{N}$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a set. We denote by $\partial\mathcal{S}$ the boundary of $\mathcal{S}$, by $\mathrm{int}(\mathcal{S})$ its interior and by $\mathrm{cl}(\mathcal{S})$ its closure. For any real $\lambda \geq 0$, the set $\lambda\mathcal{S}$ is defined as $\{x \in \mathbb{R}^n : x = \lambda y,\ y \in \mathcal{S}\}$.

Consider the time-invariant discrete-time autonomous nonlinear system described by

$$x_{k+1} = G(x_k), \tag{1}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary, *possibly discontinuous*, nonlinear function. A point $x^* \in \mathbb{R}^n$ is an equilibrium point of system (1), if $G(x^*) = x^*$. For convenience we recall the following definitions related to stability.

Definition II.1 Let $x^* \in \mathbb{R}^n$ be an equilibrium point of system (1) and let $\mathcal{X} \subseteq \mathbb{R}^n$ be a set that contains an open neighborhood of x^*.

1) The equilibrium x^* is *Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|x_0 - x^*\| \leq \delta \quad \Rightarrow \quad \|x_k - x^*\| \leq \varepsilon \quad \text{for all} \quad k \geq 0,$$

where x_k is the state of system (1) at time $k \geq 0$ with initial state x_0 at time $k = 0$.

2) The equilibrium x^* is *attractive in $\mathcal{X}$* if

$$\lim_{k\to\infty} \|x_k - x^*\| = 0, \quad \text{for all} \quad x_0 \in \mathcal{X}.$$

3) The equilibrium x^* is *locally attractive* if there exists a $\delta > 0$ such that

$$\|x_0 - x^*\| \leq \delta \quad \Rightarrow \quad \lim_{k\to\infty} \|x_k - x^*\| = 0.$$

4) The equilibrium x^* is *globally attractive* if it is attractive in $\mathbb{R}^n$.
5) The equilibrium x^* is *asymptotically stable in $\mathcal{X}$ in the Lyapunov sense* if it is both Lyapunov stable and attractive in $\mathcal{X}$.
6) The equilibrium x^* is *locally (globally) asymptotically stable in the Lyapunov sense* if it is both Lyapunov stable and locally (globally) attractive.
7) The equilibrium x^* is *exponentially stable in $\mathcal{X}$* if there exist $\theta > 0$ and $\lambda \in [0, 1)$ such that

$$\|x_k - x^*\| \leq \theta \|x_0 - x^*\| \lambda^k, \quad \text{for all} \quad x_0 \in \mathcal{X} \quad \text{and for all} \quad k \geq 0.$$

8) The equilibrium x^* is *locally exponentially stable* if there exists a $\delta > 0$, $\theta > 0$ and $\lambda \in [0, 1)$ such that

$$\|x_0 - x^*\| \leq \delta \quad \Rightarrow \quad \|x_k - x^*\| \leq \theta \|x_0 - x^*\| \lambda^k, \quad \text{for all} \quad k \geq 0.$$

9) The equilibrium x^* is *globally exponentially stable* if it is exponentially stable in $\mathbb{R}^n$.

Definition II.2 A real-valued scalar function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ *belongs to class $\mathcal{M}$* ($\varphi \in \mathcal{M}$) if it is continuous, non-decreasing and if $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$.

Definition II.3 Let $0 \leq \lambda \leq 1$ be given. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a λ-*contractive set* for system (1) if for all $x \in \mathcal{P}$ it holds that $G(x) \in \lambda\mathcal{P}$. For $\lambda = 1$ a λ-contractive set is called a *positively invariant set*.

Definition II.4 A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called *the maximal positively invariant set* contained in a set $\mathcal{X} \subseteq \mathbb{R}^n$ for system (1) if the following conditions are satisfied:

1) $\mathcal{P} \subseteq \mathcal{X}$.
2) $\mathcal{P}$ is a positively invariant set for system (1).
3) If $\tilde{\mathcal{P}}$ is a positively invariant set for system (1) and $\tilde{\mathcal{P}} \subseteq \mathcal{X}$, then $\tilde{\mathcal{P}} \subseteq \mathcal{P}$.

A polyhedron is a convex set obtained as the intersection of a finite number of open and/or closed half-spaces. Moreover, a convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a C-set [17]. A piecewise polyhedral set is a finite union of polyhedral sets. The p-norm of a vector $x \in \mathbb{R}^n$ is defined as:

$$\|x\|_p \triangleq \begin{cases} (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{i=1,\ldots,n} |x_i|, & p = \infty \end{cases},$$

where x_i, $i = 1, \ldots, n$ is the *i*-th component of x. For a matrix $Z \in \mathbb{R}^{m \times n}$ we define

$$\|Z\|_p \triangleq \sup_{x \neq 0} \frac{\|Zx\|_p}{\|x\|_p}, \quad p \geq 1,$$

as the induced matrix norm. It is well known [18] that $\|Z\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |Z^{\{ij\}}|$, where $Z^{\{ij\}}$ is the *ij*-th entry of Z. For a matrix $Z \in \mathbb{R}^{m \times n}$ with full-column rank, $Z^{-L} := (Z^\top Z)^{-1} Z^\top$ denotes the Moore-Penrose inverse [18], which satisfies $Z^{-L} Z = I_n$. For a positive definite matrix Z, $Z^{\frac{1}{2}}$ denotes the Cholesky factor [18], which satisfies $(Z^{\frac{1}{2}})^\top Z^{\frac{1}{2}} = Z^{\frac{1}{2}} (Z^{\frac{1}{2}})^\top = Z$ and, $\lambda_{\min}(Z)$ and $\lambda_{\max}(Z)$ denote the smallest and the largest eigenvalue of Z, respectively.

III. PROBLEM STATEMENT

Consider the time-invariant discrete-time nonlinear system

$$x_{k+1} = g(x_k, u_k), \tag{2}$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input at the discrete-time instant $k \geq 0$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary, *possibly discontinuous*, nonlinear function. The sets $\mathbb{X}$ and $\mathbb{U}$ specify state and input constraints and it is assumed that they are polyhedral C-sets. We assume for simplicity that the origin is an equilibrium state for (2) with $u = 0$, meaning that $g(0, 0) = 0$. Note that the class of nonlinear dynamical systems (2) contains certain classes of hybrid systems, such as PWA systems, due to the fact that g may be discontinuous. For a fixed

$N \in \mathbb{N}$, $N \geq 1$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) := (x_{k+1}, \ldots, x_{k+N})$ denote a state sequence generated by system (2) from initial state x_k and by applying the input sequence $\mathbf{u}_k := (u_k, \ldots, u_{k+N-1}) \in \mathbb{U}^N$. Furthermore, let $\mathcal{X}_T \subseteq \mathbb{X}$ denote a desired target set that contains the origin.

Definition III.1 The class of *admissible input sequences* defined with respect to $\mathcal{X}_T$ and state $x_k \in \mathbb{X}$ is $\mathcal{U}_N(x_k) := \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N,\ x_{k+N} \in \mathcal{X}_T\}$.

Now consider the following constrained optimization problem.

Problem III.2 Let the target set $\mathcal{X}_T \subseteq \mathbb{X}$ and $N \geq 1$ be given and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0,0) = 0$ be mappings. At time $k \geq 0$ let $x_k \in \mathbb{X}$ be given and minimize the cost function

$$J(x_k, \mathbf{u}_k) \triangleq F(x_{k+N}) + \sum_{i=0}^{N-1} L(x_{k+i}, u_{k+i}) \tag{3}$$

over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.

In the following, we call F, L and N the terminal cost, the stage cost and the prediction horizon, respectively. We call an initial state $x \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x) \neq \emptyset$. Similarly, Problem III.2 is said to be feasible (or *solvable*) for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathcal{X}_f(N)$ denote the set of *feasible initial states* with respect to Problem III.2 and let

$$V_{\text{MPC}} : \mathcal{X}_f(N) \to \mathbb{R}_+, \quad V_{\text{MPC}}(x_k) \triangleq \inf_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k) \tag{4}$$

denote the value function corresponding to (3). Throughout the paper we assume that there exists an optimal sequence of controls

$$\mathbf{u}_k^* \triangleq (u_k^*, u_{k+1}^*, \ldots, u_{k+N-1}^*) \tag{5}$$

calculated for state $x_k \in \mathcal{X}_f(N)$ and Problem III.2. Hence, the infimum in (4) is a minimum and $V_{\text{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. The following stability analysis is not affected by the possible non-uniqueness of the optimal control sequence (5), i.e. all results apply irrespective of which optimal sequence is selected. Let $\mathbf{x}_k^*(x_k, \mathbf{u}_k^*) := (x_{k+1}^*, \ldots, x_{k+N}^*)$ denote the state sequence generated by system (2) from initial state $x_k \in \mathcal{X}_f(N)$ and by applying the optimal sequence of controls $\mathbf{u}_k^*$. Let $\mathbf{u}_k^*(1)$ denote the first element of the sequence (5). According to the receding horizon strategy, the *MPC control law* is defined as

$$u_k^{\text{MPC}} = \mathbf{u}_k^*(1); \quad k \in \mathbb{N}. \tag{6}$$

A precise problem formulation can now be stated as follows.

Problem III.3 Let a desired set of initial states $\mathbb{X}_0 \subseteq \mathbb{X}$, system (2) and the stage cost L be given. Determine a terminal cost F, a terminal constraint set $\mathcal{X}_T$ and a prediction horizon N such that system (2) in closed-loop with the MPC control (6) is asymptotically stable in the Lyapunov sense in $\mathcal{X}_f(N)$ and $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$.

Note that many of the hybrid MPC schemes only guarantee attractivity, e.g., see [1], [5–7], and not Lyapunov stability, which is an important property in practice. This is due to the fact that if attractivity alone is ensured, then in principle, an arbitrarily small perturbation from the equilibrium may cause the state of the closed-loop system to drift far away by a fixed distance before converging back to the origin.

IV. Discrete-time Lyapunov stability

In this section we formulate discrete-time stability results for the *discontinuous* autonomous nonlinear system (1). We assume that $x^* = 0$ is an equilibrium point for system (1), i.e. $G(0) = 0$, and we derive sufficient conditions for asymptotic stability and exponential stability. Consider a non-negative scalar function $V : \mathbb{R}^n \to \mathbb{R}_+$ with $V(0) = 0$ and let $\Delta V(x_k) := V(x_{k+1}) - V(x_k) = V(G(x_k)) - V(x_k)$ denote the forward difference of V. Let w, ψ and r be class $\mathcal{M}$ functions and consider the following assumptions.

Assumption IV.1 For every $\varepsilon > 0$ there exists a $\delta(\varepsilon) \in (0, \varepsilon)$ such that $\psi(\delta) < w(\varepsilon)$.

Assumption IV.2 $w(\|x\|) := a\|x\|^\sigma$, $\psi(\|x\|) := b\|x\|^\sigma$, $r(\|x\|) := c\|x\|^\sigma$ for some $a, b, c, \sigma > 0$.

Theorem IV.3 *Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a positively invariant set for system* (1) *that contains a neighborhood $\mathcal{N}$ of the equilibrium $x^* = 0$ and let w, ψ and r be class $\mathcal{M}$ functions. Suppose there exists a non-negative scalar function $V : \mathcal{X} \to \mathbb{R}_+$ with $V(0) = 0$ such that:*

$$V(x) \geq w(\|x\|), \quad \forall x \in \mathcal{X}, \tag{7a}$$

$$V(x) \leq \psi(\|x\|), \quad \forall x \in \mathcal{N}, \tag{7b}$$

$$\Delta V(x) \leq -r(\|x\|), \quad \forall x \in \mathcal{X}. \tag{7c}$$

Then the following results hold:

1) *Under Assumption IV.1 the origin of the nonlinear system* (1) *is asymptotically stable in the Lyapunov sense in* $\mathcal{X}$.
2) *Under Assumption IV.2 the origin of the nonlinear system* (1) *is locally exponentially stable. Moreover, if the inequality* (7b) *holds for* $\mathcal{N} = \mathcal{X}$, *then the origin of the nonlinear system* (1) *is exponentially stable in* $\mathcal{X}$.

Proof: *Stability*. Let x_k represent the solution of (1) at time k, obtained from the initial condition x_0 at time $k = 0$. Choose an $\eta > 0$ such that the ball $\mathcal{B}_\eta := \{x \in \mathbb{R}^n \mid \|x\| \leq \eta\}$ satisfies $\mathcal{B}_\eta \subseteq \mathcal{N}$. Due to Assumption IV.1 we can choose for any $0 < \varepsilon \leq \eta$ a $\delta \in (0, \varepsilon)$ such that $\psi(\delta) < w(\varepsilon)$. For any $x_0 \in \mathcal{B}_\delta \subseteq \mathcal{X}$, due to positive invariance of $\mathcal{X}$, from (7) and Assumption IV.1 it follows that

$$\ldots \leq V(x_{k+1}) \leq V(x_k) \leq \ldots \leq V(x_0) \leq \psi(\|x_0\|) \leq \psi(\delta) < w(\varepsilon).$$

Since from (7a) we have that $V(x) \geq w(\varepsilon)$ for all $x \in \mathcal{X} \setminus \mathcal{B}_\varepsilon$ it follows that $x_k \in \mathcal{B}_\varepsilon$ for all $k \geq 0$. Hence, the origin of the nonlinear system (1) is *Lyapunov stable*.

Attractivity. Since V is lower bounded by zero and $\Delta V(x_k) \leq 0$, it follows that $\lim_{k\to\infty} V(x_k) = V_L \geq 0$ exists. Then, $\lim_{k\to\infty} \Delta V(x_k) = V_L - V_L = 0$. Since $0 \leq r(\|x_k\|) \leq -\Delta V(x_k)$, it follows that $\lim_{k\to\infty} r(\|x_k\|) = 0$. Assume by contradiction that $\|x_k\| \nrightarrow 0$ for $k \to \infty$. Then there exists a subsequence $\{x_{k_n}\}$ such that $\|x_{k_n}\| > \mu > 0$ for all $n \geq 0$, which by monotonicity and positivity of r implies that $r(\|x_{k_n}\|) \geq r(\mu) > 0$ for all $n \geq 0$. Hence, we reached a contradiction of convergence of $r(\|x_k\|)$ to zero. Then $\lim_{k\to\infty} \|x_k\| = 0$ for all $x_0 \in \mathcal{X}$, which implies that the origin of the nonlinear system (1) is attractive in $\mathcal{X}$ and thus, we have *asymptotic stability in* $\mathcal{X}$ *in the Lyapunov sense*.

Exponential stability. Suppose $x_0 \in \mathcal{B}_\delta$. Then $x_k \in \mathcal{B}_\varepsilon \subseteq \mathcal{N}$ for all $k \in \mathbb{N}$. Therefore it holds that $V(x_k) \leq \psi(\|x_k\|)$ and $\Delta V(x_k) \leq -r(\|x_k\|)$ for all $k \in \mathbb{N}$. Then, by Assumption IV.2, we have that for all $k \in \mathbb{N}$

$$V(G(x_k)) - V(x_k) \leq -c\|x_k\|^\sigma = -\frac{c}{b}\psi(\|x_k\|) \leq -\frac{c}{b}V(x_k).$$

This implies that:

$$V(x_k) \leq (1 - \frac{c}{b})^k V(x_0) \quad \text{for all} \quad k \geq 0.$$

In order to show that $0 \leq 1 - \frac{c}{b} < 1$, we use the inequalities (7b) and (7c), which yield:

$$0 \leq V(G(x_k)) \leq V(x_k) - c\|x_k\|^\sigma \leq \psi(\|x_k\|) - c\|x_k\|^\sigma = (b - c)\|x_k\|^\sigma.$$

Hence, it follows that $b \geq c > 0$. Then, we have that $\rho := 1 - \frac{c}{b} \in [0, 1)$. From (7a), (7b) and by Assumption IV.2 it follows that

$$a\|x_k\|^\sigma \leq V(x_k) \leq \rho^k V(x_0) \leq \rho^k b \|x_0\|^\sigma, \quad \text{for all} \quad k \geq 0.$$

Hence, $\|x_k\| \leq \theta \|x_0\| \lambda^k$ for all $x_0 \in \mathcal{B}_\delta$ and all $k \geq 0$, with $\theta := (\frac{b}{a})^{\frac{1}{\sigma}} > 0$ and $\lambda := \rho^{\frac{1}{\sigma}} \in [0, 1)$. This means that the origin of the nonlinear system (1) is *locally exponentially stable*, i.e. in a ball $\mathcal{B}_\delta \subseteq \mathcal{N}$. Moreover, since $\mathcal{X}$ is a positively invariant set for system (1), if inequality (7b) holds for $\mathcal{N} = \mathcal{X}$ then, by applying the same reasoning as above, it follows that the origin of the nonlinear system (1) is *exponentially stable in* $\mathcal{X}$. ■

Remark IV.4 It is crucial to point out the following aspects regarding Theorem IV.3:

1) The hypothesis of Theorem IV.3 allows both V and G to be discontinuous for $x \neq 0$.
2) The requirement that w, ψ and r are class $\mathcal{M}$ functions replaces the more common and more restrictive requirement that w, ψ and r are class $\mathcal{K}$ functions [11] ($\mathcal{K} \subset \mathcal{M}$).
3) For $x \in \mathcal{B}_\delta \subseteq \mathcal{N}$ we have that $\|x\| \leq \delta$, which implies that for $x \in \mathcal{X} \setminus \mathcal{B}_\delta$, $\|x\| > \delta$. Then, from inequality (7a) it follows that there exists a lower bound on V outside the ball $\mathcal{B}_\delta$, i.e. for $x \in \mathcal{X} \setminus \mathcal{B}_\delta$. This replaces the more common and somewhat more restrictive assumption that V is radially unbounded (i.e. $V(x) \to \infty$ as $\|x\| \to \infty$).

The classical proof of the first result of Theorem IV.3, e.g. the one given in [10–12], is based on the fact that G is continuous. However, if one can choose δ such that Assumption IV.1 holds, then the continuity of V and G is no longer a necessary condition. In [16] this was pointed out for the particular case of PWA systems and Piecewise Quadratic (PWQ) Lyapunov functions, which is a special case of the general Theorem IV.3. Also, in [19] it was observed that V does not need to be continuous in order to achieve Lyapunov stability. Due to the fact that [19] dealt with stability of perturbed Lipschitz continuous nonlinear systems, this issue was not further pursued. Since Theorem IV.3 applies to discontinuous V and G, this is a result of considerable importance for general discontinuous dynamical systems and hybrid systems, as will be made clear in the sequel.

V. Stability of hybrid model predictive control

In this section we investigate the MPC stabilization of the *discontinuous* nonlinear system (2), which also includes certain relevant classes of hybrid systems. We will employ *terminal cost and* *constraint set* and *terminal equality constraint* methods as the ones used for *smooth* nonlinear systems in [9] in order to guarantee stability for the closed-loop system (2)-(6). Typically, these methods rely on the fact that V_{MPC} and the system dynamics are continuous (e.g., see Section 3.2 of [9] or Theorem 4.4.2 of [20]). This requirement is induced by the classical Lyapunov proof of Theorem IV.3 [10], as mentioned before. Of course, this condition is easily satisfied for (unconstrained) linear systems and *smooth* nonlinear systems by using a common MPC cost criterion. However, it no longer holds in the case of discontinuous dynamical systems and hybrid systems. Actually, in the survey [9] it was pointed out that all the concepts and ideas used in MPC should be reconsidered in the hybrid context.

A. Terminal cost and constraint set

Consider an auxiliary static state-feedback control law

$$\tilde{u}_k \triangleq h(x_k), \tag{8}$$

with h being an arbitrary, possibly discontinuous, nonlinear function which is zero at zero ($h(0) = 0$). Let $\mathcal{X}_{\mathbb{U}} := \{x \in \mathbb{X} \mid h(x) \in \mathbb{U}\}$ denote the safe set with respect to *state and input* constraints for this control law.

Assumption V.1 There exist $w, \psi \in \mathcal{M}$ such that $L(x,u) \geq w(\|x\|)$ for all $x \in \mathcal{X}_f(N)$ and all $u \in \mathbb{U}$, and $F(x) \leq \psi(\|x\|)$ for all $x \in \mathcal{X}_T$.

Theorem V.2 *Suppose $\mathcal{X}_T$ is a closed positively invariant set for the closed-loop system* (2)-(8) *that contains the origin in its interior and that $\mathcal{X}_T$ is contained in the safe set $\mathcal{X}_{\mathbb{U}}$. Fix $N \geq 1$. Furthermore, suppose that the following inequality is satisfied:*

$$F(g(x_k, h(x_k))) - F(x_k) + L(x_k, h(x_k)) \leq 0, \quad \textit{for all} \quad x_k \in \mathcal{X}_T, \tag{9}$$

where $h(x_k)$ defines the control law (8)*. Then it holds that*

1) *If Problem III.2 is feasible at time $k \in \mathbb{N}$ for state $x_k \in \mathbb{X}$, then Problem III.2 is feasible at time $k+1$ for state $x_{k+1} = g(x_k, u_k^{MPC})$. Moreover, Problem III.2 is feasible for all $x \in \mathcal{X}_T$.*
2) *Under Assumption IV.1 and Assumption V.1 the origin of system* (2) *in closed-loop with the MPC control* (6) *is asymptotically stable in $\mathcal{X}_f(N)$, while satisfying the state and input constraints.*

3) *Under Assumption IV.2 and Assumption V.1 the origin of system* (2) *in closed-loop with the MPC control* (6) *is locally exponentially stable, while satisfying the state and input constraints.*

Proof: Consider the optimal sequence of controls (5) and the shifted sequence of controls

$$\mathbf{u}_{k+1} \triangleq (u^*_{k+1}, u^*_{k+2}, \ldots, u^*_{k+N-1}, \tilde{u}_{k+N}), \tag{10}$$

where the auxiliary control $\tilde{u}_{k+N}$ denotes the control law (8) at time $k+N$.

1) If Problem III.2 is feasible at time $k \in \mathbb{N}$ for state $x_k \in \Omega_j$ then there exists $\mathbf{u}^*_k \in \mathcal{U}_N(x_k)$ that solves Problem III.2. Then it follows that $x_{k+N} \in \mathcal{X}_T$. Since $\mathcal{X}_T \subseteq \mathcal{X}_{\mathbb{U}}$ is positively invariant for system (22) it follows that $\mathbf{u}_{k+1} \in \mathcal{U}_N(x_{k+1})$. Hence, Problem III.2 is feasible for state $x_{k+1} = g(x_k, u^{\text{MPC}}_k)$. Moreover, all states in the set $\mathcal{X}_T \subseteq \mathcal{X}_{\mathbb{U}}$ are feasible with respect to Problem III.2, as the feedback (8) can be applied for any $k \geq 0$. This implies that $\mathcal{X}_T \subseteq \mathcal{X}_f(N)$.

2) From (3), (4) and by Assumption V.1 we have that

$$V_{\text{MPC}}(x_k) \geq L(x_k, u^{\text{MPC}}_k) \geq w(\|x_k\|), \quad \forall x \in \mathcal{X}_f(N). \tag{11}$$

Let $\tilde{\mathbf{x}}_k(x_k) := (\tilde{x}_{k+1}, \ldots, \tilde{x}_{k+N})$ denote the state sequence generated by the "local" dynamics $x_{k+1} = g(x_k, h(x_k))$ from initial state $x_k \in \mathcal{X}_T$. Since $\tilde{\mathbf{x}}_k(x_k) \in \mathcal{X}_T^N$, (9) holds for all elements of the sequence $\tilde{\mathbf{x}}_k(x_k)$, yielding:

$$F(\tilde{x}_{k+1}) - F(x_k) + L(x_k, h(x_k)) \leq 0, \quad F(\tilde{x}_{k+2}) - F(\tilde{x}_{k+1}) + L(\tilde{x}_{k+1}, h(\tilde{x}_{k+1})) \leq 0,$$
$$\ldots, \quad F(\tilde{x}_{k+N}) - F(\tilde{x}_{k+N-1}) + L(\tilde{x}_{k+N-1}, h(\tilde{x}_{k+N-1})) \leq 0.$$

From the above inequalities, by optimality and by Assumption V.1 it follows that

$$V_{\text{MPC}}(x_k) \leq J(x_k, \tilde{\mathbf{u}}_k) \leq F(x_k) \leq \psi(\|x_k\|), \quad \forall x_k \in \mathcal{X}_T, \tag{12}$$

where $\tilde{\mathbf{u}}_k := (h(x_k), \ldots, h(\tilde{x}_{k+N-1}))$. By optimality, we observe that for all $x_k \in \mathcal{X}_f(N)$

$$\begin{aligned} \Delta V_{\text{MPC}}(x_k) &= J(x_{k+1}, \mathbf{u}^*_{k+1}) - J(x_k, \mathbf{u}^*_k) \leq J(x_{k+1}, \mathbf{u}_{k+1}) - J(x_k, \mathbf{u}^*_k) = \\ &= -L(x_k, u^{\text{MPC}}_k) + F(\tilde{x}_{k+N+1}) - F(x^*_{k+N}) + L(x^*_{k+N}, h(x^*_{k+N})). \end{aligned} \tag{13}$$

By the hypothesis (9), from $x^*_{k+N} \in \mathcal{X}_T$ and using Assumption V.1 it follows that

$$\Delta V_{\text{MPC}}(x_k) \leq -L(x_k, u^{\text{MPC}}_k) \leq -w(\|x_k\|), \quad \forall x_k \in \mathcal{X}_f(N). \tag{14}$$

We observe that under Assumption V.1 and Assumption IV.1 V_{MPC} satisfies the hypothesis of Theorem IV.3 for the class $\mathcal{M}$ functions w, ψ, $r = w$ and for $\mathcal{X} = \mathcal{X}_f(N)$, $\mathcal{N} = \mathcal{X}_T$. Hence, the second statement of Theorem V.2 follows from Theorem IV.3.

3) From the proof of 2) it also follows that V_{MPC} satisfies the hypothesis of Theorem IV.3 for the class $\mathcal{M}$ functions w, ψ, $r = w$ and for $\mathcal{X} = \mathcal{X}_f(N)$, $\mathcal{N} = \mathcal{X}_T$. Hence, the last statement of Theorem V.2 follows from Theorem IV.3. ■

Next, consider the closed-loop nonlinear system (2)-(8), i.e.

$$x_{k+1} = g(x_k, h(x_k)). \tag{15}$$

In the sequel we will make use of the following result obtained as a by-product of Theorem V.2.

Corollary V.3 *Consider the closed-loop system* (15). *Suppose there exists a class $\mathcal{M}$ function w such that $F(x) \geq w(\|x\|)$ for all $x \in \mathcal{X}_T$. Furthermore, suppose that the hypothesis of Theorem V.2 and Assumption V.1 hold. Then we have that:*

1) *Under Assumption IV.1 the origin of system* (15) *is asymptotically stable in $\mathcal{X}_T$, while satisfying the state and input constraints.*
2) *Under Assumption IV.2 and if $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^m$ and both* (9) *and Assumption V.1 hold for $\mathcal{X}_T = \mathbb{R}^n$, the origin of system* (15) *is globally exponentially stable.*

The proof readily follows from the fact that (9) implies

$$F(g(x_k, h(x_k))) - F(x_k) \leq -w(\|x_k\|) < 0, \quad \text{for all} \quad x_k \in \mathcal{X}_T \setminus \{0\} \tag{16}$$

and by using the reasoning used in the proof of Theorem V.2. It is worth pointing out that Corollary V.3 states that F is a local Lyapunov function for the closed-loop system (15).

Remark V.4 In order to solve Problem III.3, one still has to compute the terminal constraint set. It follows from Theorem V.2 that it is sufficient to take $\mathcal{X}_T$ as a positively invariant set for system (15) that contains the origin in its interior, in order to achieve stability. Depending of the class of systems, there are several methods that can be used to obtain $\mathcal{X}_T$, as will be illustrated in the next sections. Also, it follows from Corollary V.3 that the sublevel sets of the Lyapunov function F are positively invariant sets. Hence, depending of the type of terminal cost, one could take $\mathcal{X}_T$ as a suitable sublevel set of F. Once the terminal set has been calculated, one can perform a reachability analysis for system (2) in order to determine the minimum value of the prediction horizon needed to ensure that $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$.

B. Terminal equality constraint

In this subsection we consider the special case when $F(x) = 0$ for all $x \in \mathbb{X}$ and $X_T = \{0\}$, which corresponds to the *terminal equality constraint* method for guaranteeing stability in MPC [9].

Assumption V.5 There exist $w, \varphi \in \mathcal{M}$ such that $L(x, u) \geq w(\|x\|)$ for all $x \in \mathcal{X}_f(N)$ and all $u \in \mathbb{U}$. There exists a neighborhood of the origin $\mathcal{N} \subseteq \mathcal{X}_f(N)$ such that $L(x^*_{k+i}, u^*_{k+i}) \leq \varphi(\|x_k\|)$ for all $x_k \in \mathcal{N}$ and $i = 0, \ldots, N-1$, where $(u^*_k, \ldots, u^*_{k+N-1})$ is an optimal sequence of controls obtained as in (5) for state $x^*_k := x_k$ and $(x^*_k, \ldots, x^*_{k+N-1})$ is the corresponding state trajectory.

Remark V.6 Assumption V.5 requires that $\mathcal{X}_f(N)$ contains the origin in its interior. This is not strictly necessary as the second condition of Assumption V.5 only needs to be satisfied for $\mathcal{N} \cap \mathcal{X}_f(N)$. However, the case when $\mathcal{X}_f(N)$ does not contain the origin in its interior requires a modification to the stability notions as the closed-loop system (2)-(6) is not defined on a neighborhood around the origin. However, the modifications are straightforward.

Theorem V.7 *Consider the closed-loop system* (2)-(6), *the MPC Problem III.2 with* $\mathcal{X}_T = \{0\}$, $F(x) = 0$ *for all* $x \in \mathbb{X}$ *and fix* $N \geq 1$. *Then it holds that*

1) *If Problem III.2 is feasible at time* $k \in \mathbb{N}$ *for state* $x_k \in \mathbb{X}$, *then Problem III.2 is feasible at time* $k+1$ *for state* $x_{k+1} = g(x_k, u_k^{MPC})$.
2) *Under Assumption IV.1 and Assumption V.5 the origin of a system* (2) *in closed-loop with the MPC control* (6) *is asymptotically stable in* $\mathcal{X}_f(N)$, *while satisfying the state and input constraints.*
3) *Under Assumption IV.2 and Assumption V.5 the origin of a system* (2) *in closed-loop with the MPC control* (6) *is locally exponentially stable, while satisfying the state and input constraints.*

Proof: The proof of the first statement of Theorem V.2 also applies to Proposition V.7 for $h(x) = 0$ for all $x \in \mathbb{X}$ and $\mathcal{X}_T = \{0\}$, which is positively invariant. By Assumption V.5, inequality (11) holds. Since $\mathcal{X}_T = \{0\}$, $F(x) = 0$ and $h(x) = 0$ for all $x \in \mathbb{X}$ the inequality

(14) holds. However, note that contrary to the proof of Theorem V.2, the terminal cost no longer provides a suitable upper bound for the value function (4). Letting $x_k^* := x_k$, by Assumption V.5 we have that

$$V_{\text{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*) = \sum_{i=0}^{N-1} L(x_{k+i}^*, u_{k+i}^*) \leq N\varphi(\|x_k\|), \quad \forall x_k \in \mathcal{N}. \tag{17}$$

We observe that under Assumption V.5 and Assumption IV.1 or Assumption IV.2 V_{MPC} satisfies the hypothesis of Theorem IV.3 for the class $\mathcal{M}$ functions w, $\psi = N\varphi$, $r = w$ and for $\mathcal{X} = \mathcal{X}_f(N)$. Hence, the last two statements follow from Theorem IV.3. ■

Remark V.8 If there exists a class $\mathcal{M}$ function ψ such that $V_{\text{MPC}}(x_k) \leq \psi(x_k)$ for all $x_k \in \mathcal{X}_f(N)$, then system (2) in closed-loop with the MPC control (6) is exponentially stable in $\mathcal{X}_f(N)$, for both terminal equality constraint and terminal cost and constraint set methods. However, the existence of a class $\mathcal{M}$ upper bound on V_{MPC} for the whole set of feasible states cannot be guaranteed in general. For example, in the terminal cost and constraint set case the terminal cost function provides a suitable upper bound only for $x_k \in \mathcal{X}_T$, due to the input constraints. Exponential stability in $\mathcal{X}_f(N)$ can be achieved if Assumption V.5 holds for all $x \in \mathcal{X}_f(N)$ and $F(x_{k+N}^*) \leq \varphi_N(\|x_k\|)$ for some $\varphi_N(\|x_k\|) \in \mathcal{M}$, which ultimately yields a suitable upper bound for V_{MPC} on $\mathcal{X}_f(N)$.

In the following sections we consider the specific cases when the cost functions F and L are defined using either quadratic forms or ∞-norms. We also provide solutions to the following problems for the class of constrained PWA systems [2].

Problem V.9

1) P1: Let the system (2) and stage cost L be given. For the terminal cost and constraint set method determine the terminal cost F and the auxiliary control law (8) such that Assumption V.1 holds and inequality (9) is satisfied for the closed-loop system (15). For the terminal equality constraint method prove that L satisfies Assumption V.5.
2) P2: Calculate a positively invariant set $\mathcal{X}_T$ for system (15) (with the feedback control law $u_k = h(x)$ obtained by solving problem P1) that contains the origin in its interior and that is contained in the safe set $\mathcal{X}_{\mathbb{U}}$.

3) P3: Given a desired set of initial conditions $\mathbb{X}_0 \subseteq \mathbb{X}$, take the terminal constraint set equal to the set obtained by solving problem P2 and calculate the minimum value of the prediction horizon N such that $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$.

Solving the above problems yields a complete solution to Problem III.3.

VI. Terminal cost and constraint set: Quadratic forms

Throughout the rest of the paper we consider the class of time-invariant discrete-time Piecewise Affine (PWA) systems [2] described by equations of the form

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{when} \quad x_k \in \Omega_j, \tag{18}$$

which is a sub-class of the discontinuous nonlinear system (2). Also, we take the auxiliary controller (8) as a PWL state-feedback control law, i.e.

$$\tilde{u}_k = h(x_k) \triangleq K_j x_k \quad \text{when} \quad x_k \in \Omega_j, \quad j \in \mathcal{S}. \tag{19}$$

Here, $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input at the discrete-time instant $k \geq 0$. $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $K_j \in \mathbb{R}^{m\times n}$, $j \in \mathcal{S}$ with $\mathcal{S} := \{1, 2, \ldots, s\}$ a *finite set* of indices and s denoting the number of discrete modes. Here, $f_j \in \mathbb{R}^n$ denotes a fixed offset vector for all $j \in \mathcal{S}$. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 := \{j \in \mathcal{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 := \{j \in \mathcal{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$, so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that the origin is an equilibrium state for (18) with $u = 0$ and we require that

$$f_j = 0 \text{ for all } j \in \mathcal{S}_0. \tag{20}$$

The class of hybrid systems described by (18)-(20) contains PWA systems which *may be discontinuous over the boundaries* and which are Piecewise Linear (PWL), instead of PWA, in the state space region $\cup_{j\in\mathcal{S}_0}\Omega_j$.

In this section we consider the case when quadratic forms are used to define the cost function, i.e. $F(x) = \|P_j^{\frac{1}{2}}x\|_2^2 = x^\top P_j x$ when $x \in \mathcal{X}_T \cap \Omega_j$ and $L(x,u) = \|Q^{\frac{1}{2}}x\|_2^2 + \|R^{\frac{1}{2}}u\|_2^2 = x^\top Q x + u^\top R u$, and we assume that $\mathcal{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ in order to obtain a solution to problem P1. This yields the following cost:

$$J(x_k, \mathbf{u}_k) \triangleq x_{k+N}^\top P_j x_{k+N} + \sum_{i=0}^{N-1} x_{k+i}^\top Q x_{k+i} + u_{k+i}^\top R u_{k+i} \quad \text{when} \quad x_{k+N} \in \Omega_j, \quad j \in \mathcal{S}_0. \tag{21}$$

In this case $P_j, Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are assumed to be positive definite matrices. From (21) it follows that

$$L(x,u) \geq x^\top Q x \geq \lambda_{\min}(Q)\|x\|_2^2$$

and that

$$F(x) \leq \max_{j\in\mathcal{S}_0} \lambda_{\max}(P_j)\|x\|_2^2.$$

Then we have that the quadratic forms based terminal cost and stage cost satisfy Assumption V.1 for $w(\|x\|) = \lambda_{\min}(Q)\|x\|_2^2$, $\psi(\|x\|) = \max_{j\in\mathcal{S}_0} \lambda_{\max}(P_j)\|x\|_2^2$, which satisfy Assumption IV.1 and Assumption IV.2 (e.g. Assumption IV.1 is satisfied for $\delta(\varepsilon) = \eta(\frac{\lambda_{\min}(Q)}{\max_{j\in\mathcal{S}_0}\lambda_{\max}(P_j)})^{\frac{1}{2}}\varepsilon$, where $\eta \in (0,1)$ ensures that $\delta(\varepsilon) < \varepsilon$).

Hence, we have shown that Assumption V.1 applies for quadratic forms based hybrid MPC. In the sequel we provide a method for calculating the terminal cost F and the auxiliary control (19) such that inequality (9) is satisfied for the PWA system (18).

A. *Computation of the terminal weight(s) - Problem P1*

Let $\mathcal{Q}_{ji} := \{x \in \Omega_j \mid \exists u \in \mathbb{U} : A_j x + B_j u + f_j \in \Omega_i\}$, $(j,i) \in \mathcal{S}_0 \times \mathcal{S}_0$ and let $\mathcal{S}_{t0} := \{(j,i) \in \mathcal{S}_0 \times \mathcal{S}_0 \mid \mathcal{Q}_{ji} \neq \emptyset\}$. The set of pairs of indices $\mathcal{S}_{t0}$ can be easily determined off-line by solving s_0^2 linear programs. Consider now the PWL sub-system of the PWA system (18), i.e.

$$x_{k+1} = A_j x_k + B_j u_k, \quad \text{when} \quad x_k \in \mathcal{X}_T \cap \Omega_j, \quad j \in \mathcal{S}_0. \tag{22}$$

The set $\mathcal{S}_{t0}$ contains all discrete mode transitions that can occur in system (22), i.e. a transition from Ω_j to Ω_i can occur if and only if $(j,i) \in \mathcal{S}_{t0}$. Letting u_k be the control law (19) in (22) and substituting the resulting closed-loop system and F in (9) yields that it is sufficient to find (P_j, K_j) with P_j positive definite for all $j \in \mathcal{S}_0$ that satisfy the matrix inequality

$$P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j > 0, \quad \forall (j,i) \in \mathcal{S}_{t0}, \tag{23}$$

for (9) to be satisfied with strict inequality. Next, we present three methods that can be used to solve the nonlinear matrix inequality (23) efficiently using semi-definite programming.

Lemma VI.1 *Let $\{(P_j, K_j, Z_j, Y_j, G_j) \mid j \in \mathcal{S}_0\}$ with Z_j,P_j positive definite and G_j invertible for all $j \in \mathcal{S}_0$ denote unknown variables that are related according to $Z_j = P_j^{-1}$, $Y_j = K_j P_j^{-1}$ and*

$K_j = Y_j G_j^{-1}$, $j \in \mathcal{S}_0$. *Then the following matrix inequalities are equivalent:*

$$\begin{pmatrix} P_j & 0 \\ 0 & P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j \end{pmatrix} > 0, \quad \forall (j,i) \in \mathcal{S}_{t0}; \tag{24}$$

$$\begin{pmatrix} Z_j & Z_j & Y_j^\top & (A_j Z_j + B_j Y_j)^\top \\ Z_j & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_j Z_j + B_j Y_j) & 0 & 0 & Z_i \end{pmatrix} > 0, \quad \forall (j,i) \in \mathcal{S}_{t0}; \tag{25}$$

$$\begin{pmatrix} Z_j & (A_j Z_j + B_j Y_j)^\top & (R^{\frac{1}{2}} Y_j)^\top & (Q^{\frac{1}{2}} Z_j)^\top \\ (A_j Z_j + B_j Y_j) & Z_i & 0 & 0 \\ R^{\frac{1}{2}} Y_j & 0 & I & 0 \\ Q^{\frac{1}{2}} Z_j & 0 & 0 & I \end{pmatrix} > 0, \quad \forall (j,i) \in \mathcal{S}_{t0}; \tag{26}$$

$$\begin{pmatrix} G_j + G_j^\top - Z_j & G_j^\top & Y_j^\top & (A_j G_j + B_j Y_j)^\top \\ G_j & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_j G_j + B_j Y_j) & 0 & 0 & Z_i \end{pmatrix} > 0, \quad \forall (j,i) \in \mathcal{S}_{t0}. \tag{27}$$

The proof of Lemma VI.1 is given in the Appendix. After solving any of the above LMIs, the terminal weights P_j and the feedbacks K_j are simply recovered as $P_j := Z_j^{-1}$ and $K_j := Y_j Z_j^{-1}$, $j \in \mathcal{S}_0$ for (25) and (26) and as $P_j := Z_j^{-1}$ and $K_j := Y_j G_j^{-1}$, $j \in \mathcal{S}_0$ for (27).

If any of the above LMIs is feasible for $P_j = P$ for all $j \in \mathcal{S}_0$ implies that $F(x) = x^\top P x$ is a local *common quadratic Lyapunov function* of the closed-loop system (22)-(19). Letting $P_j \neq P_i$ for $i \neq j$, $(i,j) \in \mathcal{S}_{t0}$ implies a relaxation in the sense that solving any of the above LMIs now amounts to searching for a *Piecewise Quadratic (PWQ) Lyapunov function* [15], [16].

Remark VI.2 In [21] and [13] some preliminary results for the terminal cost and constraint set method for hybrid MPC based on quadratic cost have been presented. The result of [21] uses (25) in order to guarantee stability for unconstrained PWL systems in closed-loop with MPC controllers. The result of [13] uses (26) and relies on [9] (where continuity of V_{MPC} is used) in order to guarantee stability of PWA systems in closed-loop with MPC controllers.

Next, we employ an S-procedure technique with respect to the matrix inequality (23), as done in [15], to further reduce conservativeness, i.e. we consider the inequality

$$P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j - E_{ji}^\top U_{ji} E_{ji} > 0, \quad \forall (j,i) \in \mathcal{S}_{t0} \tag{28}$$

in the unknowns (P_j, K_j, U_{ji}), where the matrices P_j are the terminal weights employed in cost (21), the matrices U_{ji} have all entries non-negative and the matrices E_{ji} define the cones $\mathcal{C}_{ji}$, which are such that $\mathcal{C}_{ji} := \{x \in \mathbb{R}^n \mid E_{ji} x \geq 0\}$ and $\mathcal{Q}_{ji} \subseteq \mathcal{C}_{ji}$ for all $(j,i) \in \mathcal{S}_{t0}$. Note that if (P_j, K_j, U_{ji}) with $P_j > 0$ and U_{ji} with all entries non-negative for all $(j,i) \in \mathcal{S}_{t0}$ satisfy (28), then it follows that

$$x^\top (P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j) x \geq x^\top (E_{ji}^\top U_{ji} E_{ji}) x \geq 0 \tag{29}$$

whenever $x \in \mathcal{Q}_{ji} \subseteq \mathcal{C}_{ji}$, $(j,i) \in \mathcal{S}_{t0}$. Hence, (9) is satisfied and conservativeness is reduced when comparing to the matrix inequality (23). However, the techniques used in the proof of Lemma VI.1 can not be used to transform (28) into an LMI, as this would require the matrices U_{ji} to be positive definite, which increases conservativeness.

We therefore develop an alternative method for finding a solution to the matrix inequality (28). This method is based on solving a sequence of LMIs that is obtained by fixing a suitable basis of the state space and successively selecting tuning parameters. Consider an eigenvalue decomposition of the terminal weight matrices from cost (21), i.e. $P_j = V_j \Sigma_j V_j^\top$, $j \in \mathcal{S}_0$ where $\Sigma_j = \operatorname{diag}(\sigma_{1j}, \ldots, \sigma_{nj})$, $\sigma_{1j} \geq \ldots \geq \sigma_{nj}$ and $V_j^\top = V_j^{-1}$. In the sequel we assume that the orthonormal matrices $\{V_j \mid j \in \mathcal{S}_0\}$ are known and let $\Gamma_j := \operatorname{diag}(\gamma_{1j}, \ldots, \gamma_{nj}), j \in \mathcal{S}_0$ denote an arbitrary diagonal matrix. Consider now the following LMI:

$$\Delta_{ji} \triangleq \begin{pmatrix} V_j \Sigma_j V_j^\top - Q - E_{ji}^\top U_{ji} E_{ji} & (A_j + B_j K_j)^\top V_i & K_j^\top \\ V_i^\top (A_j + B_j K_j) & \Gamma_i & 0 \\ K_j & 0 & R^{-1} \end{pmatrix} > 0, \quad \forall (j,i) \in \mathcal{S}_{t0}, \tag{30}$$

in the unknowns $\{(\sigma_{1j}, \ldots, \sigma_{nj}), (\gamma_{1i}, \ldots, \gamma_{ni}), K_j, U_{ji} \mid (j,i) \in \mathcal{S}_{t0}\}$. In addition to (30) we require that the linear scalar inequalities

$$\sigma_{1j} \geq \ldots \geq \sigma_{nj} > 0, \quad \gamma_{nj} \geq \ldots \geq \gamma_{1j} > 0, \tag{31a}$$

$$\frac{1}{\epsilon_{lj}} - \sigma_{lj} \geq 0, \quad \epsilon_{lj} - \gamma_{lj} \geq 0, \quad l = 1, \ldots, n, \tag{31b}$$

with ϵ_{lj} fixed constants (scaling factors) in $(0, 1]$, are satisfied for all $j \in \mathcal{S}_0$ and that

$$U_{ji} \text{ has all entries non-negative } , \forall (j,i) \in \mathcal{S}_{t0}. \tag{32}$$

Note that the scaling factors $\epsilon_{lj} \in (0, 1]$ are assumed to be known in (31) and that condition (32) can be easily written as an LMI. Hence, the conditions (30)-(31)-(32) are in the LMI form.

Theorem VI.3 *Choose the orthonormal matrices V_j and the scaling factors $\epsilon_{lj} \in (0, 1]$, $l = 1, \dots, n$, $j \in \mathcal{S}_0$ such that the LMI* (30)-(31)-(32) *is feasible. Let $(\sigma_{1j}, \dots, \sigma_{nj})$, $(\gamma_{1i}, \dots, \gamma_{ni})$, K_j, U_{ji} be a solution. Then (P_j, K_j, U_{ji}) with $P_j = V_j \operatorname{diag}(\sigma_{1j}, \dots, \sigma_{nj}) V_j^\top > 0$ is a solution of the matrix inequality* (28).

The proof of Theorem VI.3 is given in the Appendix. Note that solving the LMI (30)-(31)-(32) hinges on the fact that the orthonormal matrices V_j and the scaling factors $\epsilon_{lj} \in (0, 1]$, $l = 1, \dots, n$, $j \in \mathcal{S}_0$ must be chosen a priori. This is not a problem with respect to the scaling factors, which can be chosen arbitrarily small. However, when it comes to fixing the matrices V_j, it is interesting to find out how they should be chosen such that by varying $\sigma_{1j}, \dots, \sigma_{nj}$ a sufficiently wide range of P_j matrices is covered. An answer to this question can be obtained for the two dimensional case, where all orthonormal matrices can be parameterized according to

$$V_j := \begin{pmatrix} -\sin\theta_j & \cos\theta_j \\ \cos\theta_j & \sin\theta_j \end{pmatrix}, \tag{33}$$

where $0 \le \theta_j \le \pi$. In this way, multiple solutions of the LMI (30)-(31)-(32) can be obtained by varying θ_j, as will be illustrated in Example 2. A similar explicit form of V_j can be specified also in the three dimensional case, by using two angles, i.e., θ_{1j} and θ_{2j}. However, these expressions get more complicated in higher dimensional spaces.

B. Computation of the terminal constraint set - Problem P2

A solution to problem P2 has been presented recently in [22], where the standard algorithm for the calculation of the maximal positively invariant set for a linear system [17], [23] has been extended to PWA systems. However, the worst-case number of one-step controllable sets that have to be calculated in the i-th iteration of the algorithm of [22] equals s_0^i, where s_0 is the number of elements of the set $\mathcal{S}_0$. Hence, this approach may lead to a combinatorial explosion of possibilities and consequently, to numerical difficulties on one hand and a complex representation of the terminal set on the other. This means that additional Boolean variables and inequalities must be added to the Problem III.2.

In this subsection we develop two methods for solving problem P2, which do not suffer from a combinatorial drawback and yield a simpler representation of the terminal set. Consider the closed-loop system (22) with the feedback gains calculated as in Section VI-A, i.e.

$$x_{k+1} = (A_j + B_j K_j)x_k =: A_j^{cl} x_k \quad \text{when} \quad x_k \in \Omega_j, \quad j \in \mathcal{S}_0. \tag{34}$$

The first method deals with the computation of a *polyhedral* positively invariant set for the PWL system (34). To do so, we consider the autonomous switched linear system corresponding to (34), i.e.

$$x_{k+1} = A_j^{cl} x_k, \quad j \in \mathcal{S}_0, \tag{35}$$

where we removed the switching rule from (34), turning the PWL system (34) into a switched linear system (35) with arbitrary switching.

Definition VI.4 Let $0 \leq \lambda \leq 1$ be given. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a λ*-contractive set* for system (35) with arbitrary switching if for all $x \in \mathcal{P}$ and all $j \in \mathcal{S}_0$ it holds that $A_j^{cl} x \in \mathcal{P}$. For $\lambda = 1$, $\mathcal{P}$ is called a *positively invariant set* for system (35) with arbitrary switching.

We make use of the following result.

Lemma VI.5 *A set which is positively invariant (λ-contractive) for the switched linear system* (35) *under arbitrary switching is also a positively invariant (λ-contractive) set for the PWL system* (34).

Proof: This follows directly from the fact that, for the PWL system (34), $x_{k+1} = A_j^{cl} x_k$ for at least one $j \in \mathcal{S}_0$ at any discrete-time instant $k \in \mathbb{N}$. ■

Since we require that $\mathcal{X}_T \subseteq \mathcal{X}_{\mathbb{U}} \cap \{\cup_{j\in\mathcal{S}_0} \Omega_j\}$ and $\mathcal{X}_{\mathbb{U}}$ is not convex in general, we consider in the following a new safe set, $\tilde{\mathcal{X}}_{\mathbb{U}}$, taken as a reasonably large polyhedral set (that contains the origin in its interior) inside $\mathcal{X}_{\mathbb{U}} \cap \{\cup_{j\in\mathcal{S}_0} \Omega_j\}$. For instance, if $\mathcal{X}_{\mathbb{U}} \subseteq \cup_{j\in\mathcal{S}_0} \Omega_j$ is a polyhedron, we set $\tilde{\mathcal{X}}_{\mathbb{U}} = \mathcal{X}_{\mathbb{U}}$ or, if $\cup_{j\in\mathcal{S}_0} \Omega_j$ is a polyhedron we could set $\tilde{\mathcal{X}}_{\mathbb{U}} = \{x \in \cup_{j\in\mathcal{S}_0} \Omega_j \mid K_j x \in \mathbb{U}, \forall j \in \mathcal{S}_0\}$. For an arbitrary target set $\mathcal{X}$ we denote $\mathcal{Q}_j^1(\mathcal{X}) := \{x \in \mathbb{R}^n \mid A_j^{cl} x \in \mathcal{X}\}$. Note that if $\mathcal{X}$ is a polyhedron that contains the origin, then $\mathcal{Q}_j^1(\mathcal{X})$ has the same properties [17].

Consider now the following sequence of sets:

$$\mathcal{X}_0 = \tilde{\mathcal{X}}_{\mathbb{U}},\ \mathcal{X}_i = \bigcap_{j\in\mathcal{S}_0} \mathcal{X}_i^j,\ i = 1, 2, \ldots, \tag{36}$$

where $\mathcal{X}_i^j := \mathcal{Q}_j^1(\mathcal{X}_{i-1}) \bigcap \mathcal{X}_{i-1},\ i = 1, 2, \ldots$.

Theorem VI.6 *The following properties hold with respect to the sequence of sets* (36)*:*

1) *The maximal positively invariant set contained in the safe set $\tilde{\mathcal{X}}_{\mathbb{U}}$ for system* (35) *with arbitrary switching is a convex set that contains the origin and is given by*
$$\mathcal{P} = \bigcap_{i=0}^{\infty} \mathcal{X}_i = \lim_{i\to\infty} \mathcal{X}_i. \tag{37}$$
2) *If an algorithm based on the recurrent sequence of sets* (36) *terminates in a finite number of iterations then the set $\mathcal{P}$ defined as in* (37) *is a polyhedral set.*
3) *If there exists a λ-contractive set with $0 < \lambda < 1$ for system* (35) *under arbitrary switching and if this set contains the origin in its interior, then an algorithm based on the recurrent sequence of sets* (36) *terminates in a finite number of iterations.*
4) *The set $\mathcal{P}$ defined as in* (37) *is a positively invariant set for the PWL system* (34).

The proof of Theorem VI.6 is given in the Appendix. If an algorithm based on (36) is used to calculate a positively invariant for system (34), then a number of s_0 one-step controllable sets $\mathcal{Q}_j^1(\mathcal{X}_{i-1})$ must be computed at each iteration, while the algorithm of [22] requires the computation of s_0^i one-step controllable sets at the i-th iteration. Hence, we have overcome the combinatorial drawback. Moreover, $\mathcal{P}$ is directly given by a finite number of linear inequalities. Thus, no additional Boolean variables need to be added for representing the terminal constraint set in Problem III.2. However, in this case $\mathcal{P}$ will not be *the maximal positively invariant set* for the PWL system (34). Then a larger prediction horizon may be required for feasibility.

Under conditions (23) or the relaxed conditions (28), a λ-contractive set can be obtained by taking a sublevel set of the PWQ Lyapunov function $F(x) = x^\top P_j x$ when $x \in \Omega_j$. Next, we present a method for obtaining *piecewise polyhedral* positively invariant sets for asymptotically stable PWA systems for which there exists a PWQ Lyapunov function.

Theorem VI.7 *Consider system* (34) *and a (piecewise ellipsoidal) sublevel set of a corresponding PWQ Lyapunov function F, i.e.*

$$\mathcal{E} := \cup_{j\in\mathcal{S}_0}\mathcal{E}_j \quad \textit{with} \quad \mathcal{E}_j := \{x \in \mathcal{X}_{\mathbb{U}} \cap \Omega_j \mid F(x) \le c\}, \quad c > 0, \quad j \in \mathcal{S}_0,$$

which is contained in the safe set $\mathcal{X}_{\mathbb{U}}$*. Let* $\alpha \in (0,1)$ *be such that* $\mathcal{E}$ *is* α*-contractive. Now assume that there exist polyhedral sets* $\mathcal{P}_j$ *that satisfy* $\alpha\mathcal{E}_j \subseteq \mathcal{P}_j \subseteq \mathcal{E}_j$ *for all* $j \in \mathcal{S}_0$*. Then the piecewise polyhedral set* $\mathcal{P} := \cup_{j\in\mathcal{S}_0}\mathcal{P}_j$ *is a positively invariant set for system* (34) *and* $\mathcal{P} \subseteq \mathcal{X}_{\mathbb{U}}$*.*

Proof: From $\alpha\mathcal{E}_j \subseteq \mathcal{P}_j \subseteq \mathcal{E}_j$ for all $j \in \mathcal{S}_0$ we have that $\alpha\mathcal{E} \subseteq \mathcal{P} \subseteq \mathcal{E}$. Thus, $\mathcal{P} \subseteq \mathcal{X}_{\mathbb{U}}$. Let $x \in \mathcal{P}$. Hence, there exists $j \in \mathcal{S}_0$ such that $x \in \mathcal{P}_j \subseteq \Omega_j$. Take $\gamma_j > 1$ such that $\gamma_j x \in \partial\mathcal{E}_j$. Then, it follows that $A_j^{cl}(\gamma_j x) \in \alpha\mathcal{E}$. Then, because of positive homogeneity of PWL dynamics, it follows that $A_j^{cl}x \in \frac{\alpha}{\gamma_j}\mathcal{E} \subseteq \alpha\mathcal{E}$. Since $\alpha\mathcal{E} \subseteq \mathcal{P}$, $\mathcal{P}$ is a positively invariant set for system (34).
■

The approach of Theorem VI.7 amounts to solving the problem of fitting a polyhedron in between two closed ellipsoidal sets where one is contained in the interior of the other. A possible way to solve this problem has been recently developed in [24] in the context of DC programming (difference of convex functions). Here, a polyhedral set is constructed by treating the ellipsoidal sets as sublevel sets of convex functions, and by exploiting upper and lower piecewise affine bounds on such functions. Giving additional structure to the algorithm of [24] such that it generates a polyhedron with a finite number of facets for each region Ω_j, a piecewise polyhedral positively invariant set is obtained for the PWL system (34). Note that this method yields a union of at most s_0 polyhedral sets, while the maximal positively invariant set computed with the algorithm of [22] may be a union of a larger number of polyhedral sets.

Another method to obtain polyhedral or piecewise polyhedral positively invariant sets for PWA systems, which is based on using ∞-norms as Lyapunov functions, will be presented in Section VII-B.

C. How to choose the prediction horizon - Problem P3

In the case of hybrid MPC based on quadratic costs, Problem III.2 with the terminal constraint set calculated as in the previous subsection leads to an MIQP problem. The minimum value of the prediction horizon N needed to ensure that $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$ can be calculated using the procedure presented in [6]. Another way to find the minimum value of the N needed for feasibility is to use the Hybrid Toolbox [25] or the MPT Toolbox [26] in order to obtain an explicit solution to Problem III.2. The explicit solution can be calculated for both quadratic forms and ∞-norms based costs (using multiparametric programming) with the Matlab function *expcon* (*mpt_control*) of the Hybrid Toolbox (MPT Toolbox), which also returns the feasible state-space region for the MPC controller, i.e. the set $\mathcal{X}_f(N)$. Thus, one can check if $\mathbb{X}_0 \subseteq \mathcal{X}_f(N)$ for a fixed N. Note that the set of feasible states for the MPC optimization problem does not depend on the type of MPC cost function (i.e. the feasible set is the same for both quadratic costs and costs based on ∞-norms).

In MPC [9], it is well known that a smaller terminal constraint set $\mathcal{X}_T$ implies that a larger N is needed for ensuring feasibility of Problem III.2. Hence, one has to make a trade-off in choosing one of the two available $\mathcal{X}_T$ sets: the maximal positively invariant set, e.g. calculated as in [22], which is represented by a possibly very large union of polyhedra, or a smaller positively invariant set, as in Theorem VI.6 (or as in Theorem VI.7), which is polyhedral (or piecewise polyhedral). Although the use of a larger terminal set obtained as in [22] may require a smaller prediction horizon for feasibility, the complexity of the resulting MPC problem still increases considerably with the number of additional Boolean variables needed to specify the terminal constraint set. The two approaches are comparable and depending on the problem at hand and the MIQP (MILP) solver one of the choices might turn out more computationally efficient.

D. Examples

The methodology developed in this section is illustrated by two examples.

Example 1. Consider the system used in [1]:

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad [1\ 0]x_k \geq 0 \\ A_2 x_k + B u_k & \text{if} \quad [1\ 0]x_k < 0 \end{cases} \tag{38}$$

subject to the constraints $x_k \in \mathbb{X} = [-5, 5] \times [-5, 5]$ and $u_k \in \mathbb{U} = [-1, 1]$, where

$$A_1 = \begin{bmatrix} 0.35 & -0.602 \\ 0.6062 & 0.35 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.35 & 0.6062 \\ -0.6062 & 0.35 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The LMI (25) has been solved for $Z_1 = Z_2 = Z$, Y_1, Y_2 and for the weights $Q = I_2$, $R = 0.4$. We have obtained the terminal weight matrix $P = \mathrm{diag}([1.4876\ 2.2434])$ and the feedback gains $K_1 = [-0.611\ -0.3572]$, $K_2 = [0.611\ -0.3572]$. We take the safe set with respect to state and input constraints as $\tilde{\mathcal{X}}_{\mathbb{U}} = \{x \in \mathbb{X} \mid |K_j x| \leq 1,\ j = 1, 2\}$. The polyhedral positively invariant set obtained with an algorithm based on the recurrent sequence of sets (36) is

$$\mathcal{X}_T = \left\{ x \in \tilde{\mathcal{X}}_{\mathbb{U}} \mid \begin{bmatrix} -0.2121 & 0.373 \\ 0.2121 & -0.373 \\ 0.2121 & 0.373 \\ -0.2121 & -0.373 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{39}$$

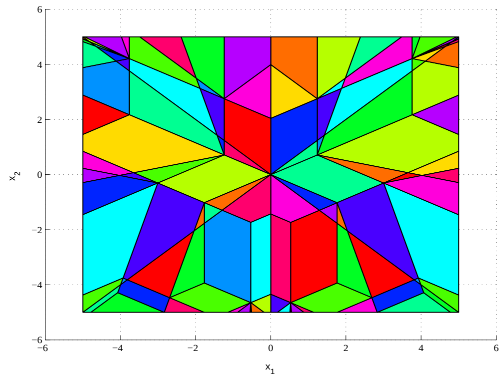

Fig. 1. Example 1: The set of feasible states obtained for $N = 4$.

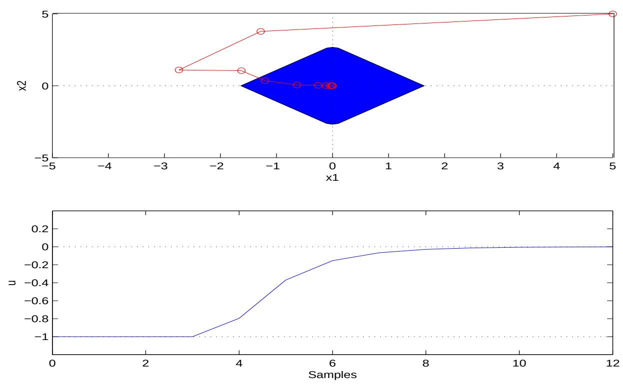

Fig. 2. Example 1: State trajectory - red; $\mathcal{X}_T$ - blue polyhedron; Input history - blue.

For system (38) and the terminal set (39), a prediction horizon of $N = 4$ is required to ensure that $\mathbb{X} \subseteq \mathcal{X}_f(N)$. The set of feasible states for $N = 4$ (obtained using the MPT Toolbox as indicated in subsection VI-C) is plotted in Figure 1. The simulation results are plotted in Figure 2 for system (38) with initial state $x_0 = [5\ 5]^\top$ in closed-loop with the MPC control (6) calculated for $N = 4$ using the Hybrid Toolbox [25], together with a plot of the terminal constraint set. For comparison purposes, we calculated the maximal positively invariant set contained in the safe set $\mathcal{X}_\mathbb{U} = \cup_{j=1,2}\{x \in \mathbb{X} \cap \Omega_j \mid |K_j x| \leq 1\}$ using the MPT Toolbox (which implements the approach of [22]). In this case the terminal set consists in the non-convex union of two polyhedra and a prediction horizon of $N = 4$ is required to ensure that $\mathbb{X} \subseteq \mathcal{X}_f(N)$.

Example 2. Consider the following open-loop unstable system:

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad E_1 x_k > 0 \\ A_2 x_k + B u_k & \text{if} \quad E_2 x_k \geq 0 \\ A_3 x_k + B u_k & \text{if} \quad E_3 x_k > 0 \\ A_4 x_k + B u_k & \text{if} \quad E_4 x_k \geq 0 \end{cases} \tag{40}$$

subject to the constraints $x_k \in \mathbb{X} = [-10, 10] \times [-10, 10]$, $u_k \in \mathbb{U} = [-1, 1]$, where

$$A_1 = \begin{bmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{bmatrix}, A_2 = \begin{bmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$A_3 = A_1$ and $A_4 = A_2$. The partitioning of the system is given by

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The weights of the MPC cost are $Q = 10^{-4} I_2$ and $R = 10^{-3}$. For system (40) the LMIs of Lemma VI.1 turn out to be infeasible. With the S-procedure approach of Section VI-A we have obtained the following solution by solving the LMI (30)-(31)-(32) for the tuning factors $\epsilon_{11} = 0.04, \epsilon_{21} = 0.3, \epsilon_{12} = 0.08, \epsilon_{22} = 1$ and for the orthonormal matrices V_1, V_2 defined as in (33) for $\theta_1 = 2.4$ and $\theta_2 = 0.9$:

$$\begin{aligned} &P_1 = \begin{bmatrix} 12.9707 & 10.9974 \\ 10.9974 & 14.9026 \end{bmatrix}, \; P_2 = \begin{bmatrix} 7.9915 & -5.5898 \\ -5.5898 & 5.3833 \end{bmatrix}, \; P_3 = P_1, \quad P_4 = P_2, \\ &K_1 = \begin{bmatrix} -0.7757 & -1.0299 \end{bmatrix}, \; K_2 = \begin{bmatrix} 0.6788 & -0.4302 \end{bmatrix}, \; K_3 = K_1, \quad K_4 = K_2, \\ &U_{11} = \begin{bmatrix} 0.4596 & 1.9626 \\ 1.9626 & 0.0198 \end{bmatrix}, \; U_{12} = \begin{bmatrix} 0.4545 & 2.0034 \\ 2.0034 & 0.0250 \end{bmatrix}, \; U_{21} = \begin{bmatrix} 0.0542 & 0.0841 \\ 0.0841 & 0.0506 \end{bmatrix}, \\ &U_{22} = \begin{bmatrix} 0.0599 & 0.0914 \\ 0.0914 & 0.0565 \end{bmatrix}, \; \sigma_{11} = 24.9765, \quad \sigma_{21} = 2.8969, \quad \sigma_{12} = 12.4273, \\ &\sigma_{22} = 0.9475, \quad \gamma_{11} = 0.0395, \quad \gamma_{21} = 0.2954, \; \gamma_{12} = 0.0791, \quad \gamma_{22} = 0.9675. \end{aligned} \tag{41}$$

A *piecewise polyhedral* positively invariant set has been computed for system (40) in closed-loop with (19) (with the feedbacks given in (41)) using the approach of Theorem VI.7 and the algorithm of [24] for the sublevel set $\mathcal{E}$ with $c = 14$, which satisfies $\mathcal{E} \subseteq \mathcal{X}_{\mathbb{U}}$. In this case $\mathcal{E}$

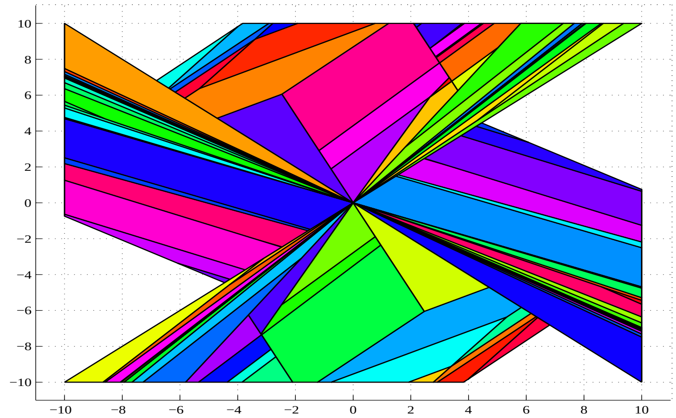

Fig. 3. Example 2: The set of feasible states obtained for $N = 4$.

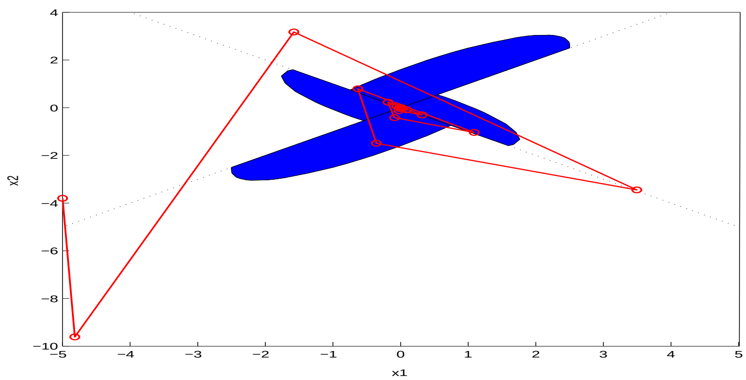

Fig. 4. Example 2: State trajectory - red; $\mathcal{X}_T$ - blue polyhedra.

is α contractive for $\alpha = 0.93$. The set of feasible states with respect to Problem III.2 obtained for system (40) with the terminal set given in Figure 4 and a prediction horizon of $N = 4$ is plotted in Figure 3. The state trajectory of system (40) with initial state $x_0 = [-5 \ \ -3.8]^\top$ and in closed-loop with the MPC control (6) calculated for $N = 4$ using the Hybrid Toolbox [25] is plotted in Figure 4. The MPC controller successfully stabilizes the open-loop unstable system (40) while satisfying the constraints.

The maximal positively invariant set calculated with the MPT Toolbox [26] (which implements the approach of [22]) for Example 2 is a non-convex union of 8 polyhedra. The resulting set of feasible states obtained for $N = 4$ in this case is comparable in size with the set plotted in Figure 3.

VII. TERMINAL COST AND CONSTRAINT SET: INFINITY NORMS

In this section we will consider the case when ∞-norms are used to define the cost function, i.e. $F(x) = \|P_j x\|_\infty$ when $x \in \mathcal{X}_T \cap \Omega_j$ and $L(x,u) = \|Qx\|_\infty + \|Ru\|_\infty$. Here $P_j \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{r\times n}$ are assumed to be matrices that have full-column rank. The MPC cost (3) now becomes:

$$J(x_k, \mathbf{u}_k) \triangleq \|P_j x_{k+N}\|_\infty + \sum_{i=0}^{N-1} \|Qx_{k+i}\|_\infty + \|Ru_{k+i}\|_\infty \quad \text{when} \quad x_{k+N} \in \Omega_j, \quad j \in \mathcal{S}. \tag{42}$$

In this setting, contrary to a quadratic forms based MPC cost, we no longer require that $\mathcal{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ in order to obtain a solution to problem P1. Also, we consider the PWA system (18), i.e.

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{when} \quad x_k \in \mathcal{X}_T \cap \Omega_j, \quad j \in \mathcal{S}, \tag{43}$$

instead of the PWL sub-system (22).

Since Q has full-column rank there always exists a positive number γ such that $\|Qx\|_\infty \geq \gamma\|x\|$ for all $x \in \mathbb{R}^n$. Then it follows that

$$L(x,u) \geq \|Qx\|_\infty \geq \gamma\|x\|_\infty, \quad \forall x \in \mathbb{R}^n, \quad \forall u \in \mathbb{R}^m.$$

For the terminal cost we have that

$$F(x) \leq \max_{j\in\mathcal{S}} \|P_j\|_\infty \|x\|_\infty, \quad \forall x \in \mathcal{X}_T.$$

Then it follows that the ∞-norms based terminal cost and stage cost satisfy Assumption V.1 for $w(\|x\|) = \gamma\|x\|_\infty$, $\psi(\|x\|) = \max_{j\in\mathcal{S}} \|P_j\|_\infty \|x\|_\infty$, which satisfy Assumption IV.1 and Assumption IV.2 (e.g. Assumption IV.1 is satisfied for $\delta(\varepsilon) = \eta\frac{\gamma}{\max_{j\in\mathcal{S}}\|P_j\|_\infty}\varepsilon$, where $\eta \in (0,1)$ ensures that $\delta(\varepsilon) < \varepsilon$).

Hence, we have shown that Assumption V.1 applies for ∞-norms based hybrid MPC. In the sequel we provide a method for calculating the terminal cost F and the auxiliary control (19) such that inequality (9) is satisfied for the PWA system (18).

A. *Computation of the terminal weight(s) - Problem P1*

Let $\mathcal{Q}_{ji} := \{x \in \Omega_j \mid \exists u \in \mathbb{U} : A_j x + B_j u + f_j \in \Omega_i\}$, $(j,i) \in \mathcal{S}\times\mathcal{S}$ and let $\mathcal{S}_t := \{(j,i) \in \mathcal{S}\times\mathcal{S} \mid \mathcal{Q}_{ji} \neq \emptyset\}$. Note that the set $\mathcal{S}_t$ defined here differs from the set $\mathcal{S}_{t0}$ defined in Section VI-A, since it also incorporates the indices $j \in \mathcal{S}_1$, i.e. $\mathcal{S}_{t0} = \mathcal{S}_t \cap \{\mathcal{S}_0 \times \mathcal{S}_0\}$. The set of pairs of indices $\mathcal{S}_t$ can be easily determined off-line by solving s^2 linear programs. The set $\mathcal{S}_t$ contains all discrete mode transitions that can occur in the PWA system (43), i.e. if $(j,i) \in \mathcal{S}_t$ then a transition from Ω_j to Ω_i can occur.

Substituting (43) and F in (9) yields that it is sufficient to find $\{(P_j, K_j) \mid j \in \mathcal{S}\}$ that satisfy:

$$\|P_i((A_j+B_jK_j)x_k+f_j)\|_\infty - \|P_jx_k\|_\infty + \|Qx_k\|_\infty + \|RK_jx_k\|_\infty \leq 0, \quad \forall x_k \in \mathcal{X}_T, \quad (j,i) \in \mathcal{S}_t, \tag{44}$$

for (9) to be satisfied. Now consider the following ∞-norm inequalities:

$$\|P_i(A_j + B_jK_j)P_j^{-L}\|_\infty + \|QP_j^{-L}\|_\infty + \|RK_jP_j^{-L}\|_\infty \leq 1 - \gamma_{ji}, \quad (j,i) \in \mathcal{S}_t \tag{45}$$

and

$$\|P_if_j\|_\infty \leq \gamma_{ji}\|P_jx\|_\infty, \quad \forall x \in \mathcal{X}_T \cap \Omega_j, \quad (j,i) \in \mathcal{S}_t, \tag{46}$$

where $\gamma_{ji} \in [0,1)$, $(j,i) \in \mathcal{S}_t$. Note that, because of (20), (46) trivially holds if $\mathcal{S} = \mathcal{S}_0$.

Theorem VII.1 *Suppose* (45)-(46) *is solvable in* (P_j, K_j, γ_{ji}) *where* P_j *has full-column rank and* $\gamma_{ji} \in [0,1)$ *for* $(j,i) \in \mathcal{S}_t$*. Then* (P_j, K_j) *with* $j \in \mathcal{S}$ *is a solution of the* ∞*-norm inequality* (44)*.*

The proof of Theorem VII.1 is given in the Appendix.

Remark VII.2 If (P_j, K_j), $j \in \mathcal{S}$ satisfy (44) it follows that

$$\|P_i(A_j + B_jK_j)x_k + P_if_j\|_\infty - \|P_jx_k\|_\infty \leq -\gamma\|x_k\|_\infty < 0, \quad \forall x_k \in \mathcal{X}_T \setminus \{0\}, \quad \forall (j,i) \in \mathcal{S}_t.$$

Hence, as indicated in Corollary V.3, the *discontinuous* function $F(x) = \|P_jx\|_\infty$ when $x \in \Omega_j$ is a local (piecewise linear) Lyapunov function for the dynamics $x_{k+1} = (A_j + B_jK_j)x_k + f_j$, $j \in \mathcal{S}$.

Finding the matrices P_j and the feedback matrices K_j that satisfy the ∞-norm inequality (45) amounts to solving off-line an optimization problem subject to the constraint $\operatorname{rank}(P_j) = n$ for all $j \in \mathcal{S}$. Note that this constraint can be replaced by the convex constraint $P_j^\top P_j > 0$. Once the matrices P_j satisfying (45) have been found, one still has to check that they also satisfy inequality (46), provided that $\mathcal{S} \neq \mathcal{S}_0$. For example, this can be verified by checking the inequality

$$\|P_if_j\|_\infty \leq \gamma_j \min_{x \in \mathcal{X}_T \cap \Omega_j} \|P_jx\|_\infty, \quad (j,i) \in \mathcal{S}_t(\mathcal{X}_T),$$

where $\mathcal{S}_t(\mathcal{X}_T) := \{(j,i) \mid \mathcal{X}_T \cap \Omega_j \neq \emptyset\} \cap \mathcal{S}_1$. In order to overcome the difficulty of solving (45)-(46) simultaneously, one can require that $\mathcal{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ is a positively invariant set only for the PWL sub-system (22), as done in Section VI for hybrid MPC based on quadratic forms. Note that the auxiliary control action (19) defines now a local state feedback, instead of a global state feedback, as in Theorem VII.1. In this case Theorem VII.1 can be reformulated as follows.

Corollary VII.3 *Suppose that the inequality*

$$\|P_i(A_j + B_jK_j)P_j^{-L}\|_\infty + \|QP_j^{-L}\|_\infty + \|RK_jP_j^{-L}\|_\infty \leq 1 \tag{47}$$

is solvable in (P_j, K_j) *for* P_j *with full-column rank for* $(j,i) \in \mathcal{S}_{t0}$ *and that* $\mathcal{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$*. Then* (P_j, K_j) *with* $j \in \mathcal{S}_0$ *is a solution of the* ∞*-norm inequality* (44).

Proof: Since $\mathcal{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ it follows that the inequality (44) only needs to be satisfied for $(j,i) \in \mathcal{S}_{t0}$, where S_{t0} is the set of indices defined in Section VI-A. From (20) we have that $f_j = 0$ for all $j \in \mathcal{S}_0$ and thus, inequality (46) is directly satisfied with equality for $\gamma_{ji} = 0$ and for all $(j,i) \in \mathcal{S}_{t0}$. Then the result follows from Theorem VII.1. ■

B. Computation of the terminal constraint set - Problem P2

From Remark VII.2 it follows that the terminal constraint set $\mathcal{X}_T$ can be simply obtained in the case of ∞-norms based hybrid MPC as

$$\mathcal{X}_T \triangleq \cup_{j\in\mathcal{S}}\{x \in \Omega_j \mid \|P_jx\|_\infty \leq \varphi^*\}, \tag{48}$$

where $\varphi^* = \sup_\varphi\{\{x \in \Omega_j \mid \|P_jx\|_\infty \leq \varphi\} \subseteq \mathcal{X}_\mathbb{U}\}$. Since this set is a finite union of polyhedra (at most a union of s polyhedra), Problem III.2 leads to an MILP problem.

Remark VII.4 The level sets of the Lyapunov function $V(x) = \|P_jx\|_\infty$ when $x \in \Omega_j$ are λ-contractive sets [17] and they are finite unions of polyhedra (i.e. they are represented by a polyhedron in each region of the PWA system). Hence, this yields a new method to obtain (in finite time) *piecewise polyhedral* λ*-contractive sets* for the class of PWA systems, which takes into account also the affine terms f_j for $j \in \mathcal{S}_1$. If we set $P_j = P$ for all $j \in \mathcal{S}$, this yields a new way to obtain *polyhedral* λ*-contractive sets* for PWA systems and switched linear systems. Note that these sets can also be used as terminal constraint sets for hybrid MPC based on a quadratic cost.

C. How to choose the prediction horizon - Problem P3

In the case of an ∞-norms based MPC cost Problem III.2 with the terminal constraint set chosen as in Section VII-B leads to an MILP problem [5]. Two procedures for obtaining the minimum prediction horizon needed to achieve feasibility of Problem III.2 have been indicated in Section VI-C.

D. Reduction of the computational complexity

This section gives some techniques to approach the computationally challenging problem associated with inequality (45). If the matrices P_j are known in (45), then the optimization problem associated with the inequality (44) can be recast as a Linear Programming (LP) problem. In the sequel we will indicate two ways to find "educated guesses" of P_j, $j \in \mathcal{S}$. These methods are based on the observation that a necessary condition for the existence of the P_j matrices that satisfy (45)-(46) is that $F(x) = \|P_j x\|_\infty$ when $x \in \Omega_j$, $j \in \mathcal{S}$ is a piecewise linear Lyapunov function of the closed-loop PWA system (43)-(19), as shown in Corollary V.3. Educated guesses of P_j are now based on functions $F(x)$ that satisfy this necessary condition and hence, induce what one might call "feedback controlled positively invariant sets".

A quadratic approach. One possibility to fix the terminal weight in (45) is to use the approach of Section VI-B to calculate a common polyhedral positively invariant set $\mathcal{P}$ for the PWL sub-system (22). If $\mathcal{P}$ is symmetric, then a good choice for the terminal weight is the matrix P that induces the polyhedron $\mathcal{P}$, i.e. $\mathcal{P} := \{x \in \mathcal{X}_{\mathbb{U}} \mid \|Px\|_\infty \leq c\}$, $c > 0$. Note that this approach towards fixing the terminal weights is using some feedback matrices $\{K_j \mid j \in \mathcal{S}_0\}$ calculated via semi-definite programming, i.e. as done in Section VI-A or in [16] in order to obtain a *common quadratic Lyapunov function*. However, these feedbacks, although they render the resulting polyhedral set positively invariant, do not necessarily satisfy the inequality (45). Fixing $P_j = P$ for all $j \in \mathcal{S}_0$ in (45) and solving the remaining LP problem in $\{K_j \mid j \in \mathcal{S}_0\}$ amounts to searching for a different state feedback control law, which not only renders the employed set positively invariant, but also ensures that inequality (45) is satisfied.

"*Squaring the circle*". Another way to obtain polyhedral (or piecewise polyhedral) controlled positively invariant sets for PWA systems that admit a common (or a piecewise) quadratic Lyapunov function is based on the result of Theorem VI.7. Giving additional structure to the algorithm of [24] such that it generates a symmetric polyhedron with a finite number of facets, a polyhedral or a piecewise polyhedral positively invariant set can be obtained for system (22) and then P_j can be chosen as the matrices that induce the corresponding polyhedra.

Example 3. Consider the following PWA chain of integrators:

$$x_{k+1} = \begin{cases} A_1 x_k + B_1 u_k & \text{if } [0\ 1\ 1]x_k < 0\ ,\ [1\ 0\ 0]x_k < 2\ ,\ [-1\ 0\ 0]x_k < 2 \\ A_2 x_k + B_2 u_k & \text{if } [0\ 1\ 1]x_k \geq 0\ ,\ [1\ 0\ 0]x_k < 2\ ,\ [-1\ 0\ 0]x_k < 2 \\ A_3 x_k + B_3 u_k + f & \text{otherwise} \end{cases} \tag{49}$$

subject to the constraints $x_k \in \mathbb{X} = [-10, 10]^3$ and $u_k \in \mathbb{U} = [-2, 2]$, where

$$A_1 = \begin{bmatrix} 1 & 0.4 & 0.08 \\ 0 & 1 & 0.4 \\ 0 & 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0.7 & 0.245 \\ 0 & 1 & 0.7 \\ 0 & 0 & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0.8 & 0.32 \\ 0 & 1 & 0.8 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0107 \\ 0.08 \\ 0.4 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.0572 \\ 0.245 \\ 0.7 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.0853 \\ 0.32 \\ 0.8 \end{bmatrix},\ f = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

The weights of the MPC cost are $Q = I_3$ and $R = 0.1$. The following solution to the inequality (45) has been found using a min-max formulation and the Matlab `fmincon` solver (CPU time was 5.65 seconds for a Pentium IV at 1.7GHz):

$$P = \begin{bmatrix} 24.1304 & 20.3234 & 4.9959 \\ 20.3764 & 35.9684 & 10.5832 \\ 6.3709 & 9.21 & 9.9118 \end{bmatrix},\ K_3 = \begin{bmatrix} -0.8434 & -2.063 & -1.9809 \end{bmatrix},\ \gamma = 0.174,$$

$$K_1 = \begin{bmatrix} -2.3843 & -4.5862 & -3.1858 \end{bmatrix},\ K_2 = \begin{bmatrix} -0.8386 & -2.1077 & -2.1084 \end{bmatrix}. \tag{50}$$

The terminal set has been obtained as in (48) for $\varphi^* = 2.64$ and is plotted in Figure 5. Due to the input constraints we have that $\mathcal{X}_T \subset \cup_{j \in \mathcal{S}_0} \Omega_j$ for system (49). However, it can be easily checked that inequality (46) holds for system (49) and all $x \in \mathbb{X}$. The simulation results are plotted in Figure 6 for system (49) with initial state $x_0 = [1.9\ \ -1\ 1]^\top$ and in closed-loop with the MPC control (6) calculated for the matrices P, Q and R given above, $N = 5$ (obtained using the Hybrid Toolbox as in subsection VII-C) and with a polyhedral terminal set (i.e. the set plotted in Figure 5). As guaranteed by Theorem V.2, the MPC control law (6) stabilizes system (49) while satisfying the state and input constraints.

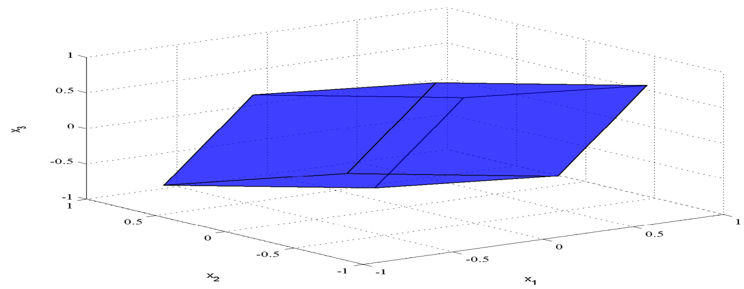

Fig. 5. Example 3: $\mathcal{X}_T$ - blue polyhedron.

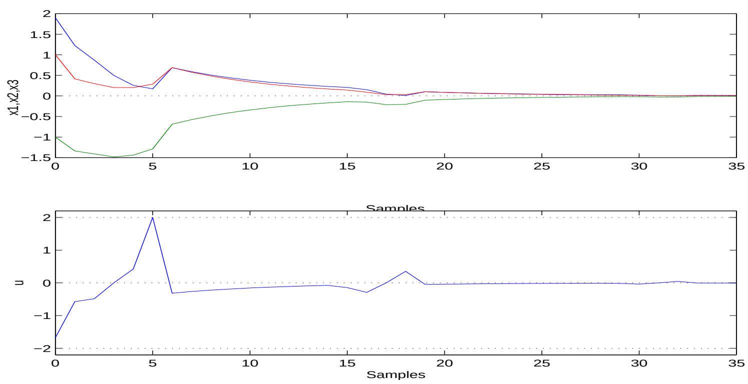

Fig. 6. Example 3: State trajectory (up) and input history (down).

The maximal positively invariant set obtained with the MPT Toolbox [26] for Example 3 is a non-convex union of 13 polyhedra. In this case, a prediction horizon of $N = 2$ is required to ensure that $x_0 \in \mathcal{X}_f(N)$.

VIII. Terminal equality constraint

In this section we consider the case when a terminal equality constraint is employed to guarantee stability, e.g. see [9] for details on this method. In this setting the terminal cost $F(x)$ is set equal to zero for all x and the terminal constraint set is taken as $\mathcal{X}_T = \{0\}$ in Problem III.2. This implies that the terminal constraint from Definition III.1 now becomes $x_{k+N} = 0$. On one hand, this method has the advantage that the problems P1 and P2 are solved directly. On the other hand, the terminal equality constraint method usually requires a larger prediction horizon for feasibility of the Problem III.2, which increases the computational complexity of the MPC algorithm (e.g., for Example 3 a prediction horizon of $N = 35$ is required for feasibility with respect to the considered initial state).

Note that the terminal equality constraint method, although it has been used since the early stages of hybrid MPC [1] has only been proven to guarantee attractivity for the closed-loop system (e.g. see Theorem 1 of [1]). We show that under suitable assumptions Lyapunov stability can also be achieved in this setting using the theory developed in Subsection V-B.

Consider an optimal sequence of controls obtained by solving Problem III.2 at time $k \geq 0$, i.e. $\mathbf{u}_k^* = (u_k^*, u_{k+1}^*, \ldots, u_{k+N-1}^*)$ and let $\mathbf{x}_k^*(x_k, \mathbf{u}_k^*) := (x_{k+1}^*, \ldots, x_{k+N}^*)$ denote the state sequence generated by system (2) from initial state x_k and by applying the input sequence $\mathbf{u}_k^*$. Note that $x_{k+N}^* = 0$. Let $\|\cdot\|_p$ denote an arbitrary p-norm and consider the following assumption.

Assumption VIII.1 There exist positive numbers β_i such that $\|u_{k+i}^*\|_p \leq \beta_i \|x_k\|_p$ for all $x_k \in \mathcal{X}_f(N)$, and all $i = 0, \ldots, N-1$.

We will use the following result.

Lemma VIII.2 *Under Assumption VIII.1 there exist positive numbers α_i such that*

$$\|x_{k+i}^*\|_p \leq \alpha_i \|x_k\|_p, \quad \text{for all} \quad x_k \in \mathcal{X}_f(N) \quad \text{and for all} \quad i = 0, \ldots, N-1. \tag{51}$$

The proof of Lemma VIII.2 is given in the Appendix. In the sequel we will show that the stage cost L satisfies Assumption V.5 for both the quadratic forms case and the p-norms case.

Theorem VIII.3 *Suppose that Assumption VIII.1 holds and $L(x,u) = x^\top Q x + u^\top R u$ or $L(x,u) = \|Qx\|_p + \|Ru\|_p$. Then the stage cost $L(x,u)$ satisfies Assumption V.5.*

Proof: We have already proven in Section VI and Section VII that L satisfies the first part of Assumption V.5 for $w(\|x\|) = \lambda_{\min}(Q)\|x\|_2^2$ in the quadratic forms case and $w(\|x\|) = \gamma \|x\|_\infty$ in the ∞-norms case. Note that the proof given in the ∞-norms case applies for any p-norm. Now we prove that the second part of Assumption V.5 is satisfied. Consider the quadratic forms stage cost, i.e. $L(x,u) = x^\top Q x + u^\top R u$. From Lemma VIII.2 and by Assumption VIII.1 it follows that:

$$\begin{aligned} L(x_{k+i}^*, u_{k+i}^*) &\leq \lambda_{\max}(Q)\|x_{k+i}^*\|_2^2 + \lambda_{\max}(R)\|u_{k+i}^*\|_2^2 \leq \\ &\leq (\alpha_i^2 \lambda_{\max}(Q) + \beta_i^2 \lambda_{\max}(R))\|x_k\|_2^2 =: c_i \|x_k\|_2^2, \quad \forall x_k \in \mathcal{X}_f(N),\ i = 0, \ldots, N-1, \end{aligned} \tag{52}$$

where $c_i > 0$ for all $i = 0, \ldots, N-1$. By applying the same reasoning for a p-norms stage cost, i.e. $L(x,u) = \|Qx\|_p + \|Ru\|_p$, it follows that:

$$\begin{aligned} L(x^*_{k+i}, u^*_{k+i}) &\leq \|Q\|_p \|x^*_{k+i}\|_p + \|R\|_p \|u^*_{k+i}\|_p \leq \\ &\leq (\alpha_i \|Q\|_p + \beta_i \|R\|_p)\|x_k\|_p =: a_i \|x_k\|_p, \quad \forall x_k \in \mathcal{X}_f(N),\ i = 0, \ldots, N-1, \end{aligned} \tag{53}$$

where $a_i > 0$ for all $i = 0, \ldots, N-1$.

Hence, the stage cost $L(x,u)$ satisfies Assumption V.5 for $w(\|x\|) = \lambda_{\min}(Q)\|x\|_2^2$ and $\varphi(\|x\|) = \max_{i=0,\ldots,N-1} c_i \|x\|_2^2$, for the quadratic forms case, and $w(\|x\|) = \gamma\|x\|_p$ and $\varphi(\|x\|) = \max_{i=0,\ldots,N-1} a_i \|x\|_p$ for the p-norms case. ■

We have shown that Assumption V.5 holds for both quadratic forms and p-norms based hybrid MPC. Hence, it follows from Theorem VIII.3 and Theorem V.7 that Lyapunov stability can be achieved for *terminal equality constraint* hybrid MPC. It is worth pointing out that in [19] it has been shown that Lyapunov stability is achieved for terminal equality constraint MPC of Lipschitz continuous nonlinear systems, based on the assumption that the controls u^*_{k+i} are Lipschitz continuous functions of the state (see Corollary 1 of [19] for details).

IX. CONCLUSIONS

In this paper we have derived sufficient *a priori* conditions for Lyapunov asymptotic stability and exponential stability of hybrid Model Predictive Control. We developed a general theory which shows that Lyapunov stability can be achieved even if the considered Lyapunov function and the system dynamics are discontinuous. This has been proven for both *terminal cost and constraint set* and *terminal equality constraint* hybrid MPC. In the particular case of constrained PWA systems and quadratic forms or ∞-norms based cost functions, new procedures for calculating the terminal cost and the terminal constraint set have been developed. If the MPC cost is defined using quadratic forms, then the terminal cost is calculated via semi-definite programming. For an ∞-norm based cost, the terminal cost is obtained by solving off-line an optimization problem. Novel algorithms for calculating polyhedral or piecewise polyhedral positively invariant sets for PWA systems have also been developed. The off-line computation of these positively invariant sets is numerically more friendly in comparison with the computation of the maximal positively invariant set. The theory has been illustrated by several examples.

In summary, next to a general theory on stability of hybrid MPC, we provide a complete framework for both quadratic forms and ∞-norms MPC schemes for PWA systems with an *a priori* stability guarantee.

APPENDIX

A. Proof of Lemma VI.1

First we prove that the matrix inequality (24) and the LMI (25) are equivalent. We start by applying the Schur complement to (25), which yields:

$$Z_j - \begin{pmatrix} Z_j & Y_j^\top & (A_jZ_j + B_jY_j)^\top \end{pmatrix} \begin{pmatrix} Q & 0 & 0 \\ 0 & R & \\ 0 & 0 & Z_i^{-1} \end{pmatrix} \begin{pmatrix} Z_j \\ Y_j \\ (A_jZ_j + B_jY_j) \end{pmatrix} > 0$$

and $\begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & R^{-1} & \\ 0 & 0 & Z_i \end{pmatrix} > 0$ for all $(j,i) \in \mathcal{S}_{t0}$. Since $Q > 0$ and $R > 0$ it follows that

$$\begin{pmatrix} Z_i & 0 \\ 0 & Z_j - Z_jQZ_j - Y_j^\top RY_j - (A_jZ_j + B_jY_j)^\top Z_i^{-1}(A_jZ_j + B_jY_j) \end{pmatrix} > 0.$$

Substituting $Z_j := P_j^{-1}$, $Z_i := P_i^{-1}$ and $Y_j := K_jP_j^{-1}$ in the above matrix inequality and pre-multiplying and post-multiplying with $\begin{pmatrix} P_i & 0 \\ 0 & P_j \end{pmatrix} > 0$ yields the equivalent matrix inequality (24).

The proof that (24) and the LMI (26) are equivalent can be obtained by applying the method used in the proof of Theorem 1 from [27] (in [27] the proof is only given for a common terminal weight P and a linear feedback K due to constraints imposed by robustness). Finally, it can be proven that (24) and the LMI (27) are equivalent by combining the technique used in the proof of Theorem 2 from [28] (which deals with the stability of feedback controlled switched linear systems) with the technique used above to prove the equivalency between (24) and (25).

B. Proof of Theorem VI.3

Since $\{(\sigma_{1j},\dots,\sigma_{nj}),(\gamma_{1i},\dots,\gamma_{ni}),K_j,U_{ji} \mid (j,i) \in \mathcal{S}_{t0}\}$ satisfy the LMI (30)-(31)-(32) we can apply the Schur complement to (30), which yields

$$V_j\Sigma_jV_j^\top - (A_j + B_jK_j)^\top V_i\Gamma_i^{-1}V_i^\top(A_j + B_jK_j) - Q - K_j^\top RK_j - E_{ji}^\top U_{ji}E_{ji} > 0.$$

By adding and subtracting $(A_j+B_jK_j)^\top V_i\Sigma_i V_i^\top(A_j+B_jK_j)$ in the above inequality we obtain the equivalent

$$\begin{aligned}V_j\Sigma_jV_j^\top - (A_j+B_jK_j)^\top V_i\Sigma_iV_i^\top(A_j+B_jK_j) - Q - K_j^\top RK_j - E_{ji}^\top U_{ji}E_{ji} > \\ > (A_j+B_jK_j)^\top V_i\Gamma_i^{-1}V_i^\top(A_j+B_jK_j) - (A_j+B_jK_j)^\top V_i\Sigma_iV_i^\top(A_j+B_jK_j). \quad (54)\end{aligned}$$

From (31b) we have that $1-\sigma_{lj}\gamma_{lj} \geq 0$ for all $l=1,\dots,n$ and all $j\in\mathcal{S}_0$. Then, the inequality

$$\Gamma_i^{-1} - \Sigma_i = \begin{pmatrix} \frac{1-\gamma_{1i}\sigma_{1i}}{\gamma_{1i}} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1-\gamma_{ni}\sigma_{ni}}{\gamma_{ni}} \end{pmatrix} \geq 0$$

holds for all $i\in\mathcal{S}_0$ and from (54) it follows that the inequality

$$V_j\Sigma_jV_j^\top - (A_j+B_jK_j)^\top V_i\Sigma_iV_i^\top(A_j+B_jK_j) - Q - K_j^\top RK_j - E_{ji}^\top U_{ji}E_{ji} > 0$$

is satisfied for all $(j,i)\in\mathcal{S}_{t0}$. The matrix inequality (28) is obtained by letting $P_j = V_j\Sigma_jV_j^\top > 0$ for all $j\in\mathcal{S}_0$ in the above inequality.

C. Proof of Theorem VI.6

1) If $x\in\mathcal{P}$ then $x\in\mathcal{X}_i$ for all i. Hence, we have that $A_j^{cl}x\in\mathcal{X}_{i-1}$ for all $j\in\mathcal{S}_0$ and all i. Then $A_j^{cl}x\in\mathcal{P}$ for all $j\in\mathcal{S}_0$. So, $\mathcal{P}$ is a positively invariant set for system (35) with arbitrary switching. In order to prove that the set $\mathcal{P}$ is maximal let $\tilde{\mathcal{P}}\subseteq\tilde{\mathcal{X}}_{\mathbb{U}}=\mathcal{X}_0$ be a positively invariant set for system (35) with arbitrary switching. In order to use induction, we assume that $\tilde{\mathcal{P}}\subseteq\mathcal{X}_i$ for some i. Due to the positive invariance of $\tilde{\mathcal{P}}$, for any $x\in\tilde{\mathcal{P}}$ we have that $A_j^{cl}x\in\tilde{\mathcal{P}}\subseteq\mathcal{X}_i$ for all $j\in\mathcal{S}_0$. Hence, $x\in\mathcal{X}_{i+1}$. Thus, $\tilde{\mathcal{P}}\subset\mathcal{X}_{i+1}$ and by induction $\tilde{\mathcal{P}}\subset\mathcal{X}_i$ for all i, which yields $\tilde{\mathcal{P}}\subset\bigcap_{i=0}^{\infty}\mathcal{X}_i=\mathcal{P}$.

Now we prove that $\mathcal{P}$ is a convex set. Assume that $\mathcal{P}$ is the maximal positively invariant set for system (35) with arbitrary switching. Then we have that $\mathcal{P}$ is a positively invariant set for any linear system in (35) and then it follows from [29] that the convex hull of $\mathcal{P}$ is also a positively invariant set for any linear system in (35). Hence, the convex hull of $\mathcal{P}$ is a positively invariant set for system (35) under arbitrary switching. Since $\tilde{\mathcal{X}}_{\mathbb{U}}$ is a convex set, it follows that the convex hull of $\mathcal{P}$ is included in $\tilde{\mathcal{X}}_{\mathbb{U}}$. By maximality, the convex hull of $\mathcal{P}$ is also included in $\mathcal{P}$ and thus, $\mathcal{P}$ is convex. As the origin is an equilibrium for $x_{k+1}=A_j^{cl}x, \forall j\in\mathcal{S}_0$, $\mathcal{P}$ contains the origin.

2) Assume that the algorithm (36) terminates in i^* steps. Then, it follows directly from $\mathcal{X}_i \subseteq \mathcal{X}_{i-1}$ for all $i > 0$ that $\mathcal{X}_i = \mathcal{X}_{i^*}$ for all $i \geq i^*$ and $\mathcal{P} = \mathcal{X}_{i^*}$. Since $\tilde{\mathcal{X}}_{\mathbb{U}}$ is a polyhedral set and from the fact that the intersection of polyhedra produces polyhedra, it follows that the sets $\mathcal{X}_0^j := \mathcal{Q}_j^1(\tilde{\mathcal{X}}_{\mathbb{U}}) \bigcap \tilde{\mathcal{X}}_{\mathbb{U}}$ are polyhedra for all $j \in \mathcal{S}_0$. Then it follows that the set $\mathcal{X}_1$ is a polyhedral set and, for the same reason, $\mathcal{X}_i$, $i = 2, 3, \ldots,$ are polyhedral sets. Then, it follows that $\mathcal{P}$ is also a polyhedral set.

3) The proof is essentially due to [23]. Let $\mathcal{E}$ denote a λ-contractive set with $0 < \lambda < 1$ for system (35) under arbitrary switching that contains the origin in its interior. Then there exist $c_2 > c_1 > 0$ such that $c_1\mathcal{E} \subsetneq \tilde{\mathcal{X}}_{\mathbb{U}} \subsetneq c_2\mathcal{E}$. Since $c_2\mathcal{E}$ is λ-contractive, we have that any state trajectory starting on the boundary of $c_2\mathcal{E}$ reaches in i discrete-time steps the set $\lambda^i c_2\mathcal{E}$. Hence, there exists an i^* such that all the states trajectories starting inside $\tilde{\mathcal{X}}_{\mathbb{U}} \subsetneq c_2\mathcal{E}$ lie in $c_1\mathcal{E}$ within i^* discrete-time steps. Since $c_1\mathcal{E}$ is λ-contractive and thus, positively invariant, it follows that if a state trajectory stays i^* discrete-time steps inside $\tilde{\mathcal{X}}_{\mathbb{U}}$, then it stays in forever. Hence, $\mathcal{X}_{i^*} \subseteq \mathcal{P}$ and thus, $\mathcal{X}_{i^*} = \mathcal{P}$.

4) This follows directly from 1) and from Lemma VI.5.

D. *Proof of Theorem VII.1*

Since $\{(P_j, K_j, \gamma_{ji}) \mid (j,i) \in \mathcal{S}_t\}$ satisfy (45) it follows that

$$\|P_i(A_j + B_jK_j)P_j^{-L}\|_\infty + \|QP_j^{-L}\|_\infty + \|RK_jP_j^{-L}\|_\infty + \gamma_{ji} - 1 \leq 0, \quad (j,i) \in \mathcal{S}_t. \tag{55}$$

Right multiplying the inequality (55) with $\|P_jx_k\|_\infty$ and using the inequality (46) yields:

$$\begin{aligned}
0 &\geq \|P_i(A_j + B_jK_j)P_j^{-L}\|_\infty\|P_jx_k\|_\infty + \|QP_j^{-L}\|_\infty\|P_jx_k\|_\infty \\
&+ \gamma_{ji}\|P_jx_k\|_\infty + \|RK_jP_j^{-L}\|_\infty\|P_jx_k\|_\infty - \|P_jx_k\|_\infty \geq \\
&\geq \|P_i(A_j + B_jK_j)P_j^{-L}P_jx_k\|_\infty + \|QP_j^{-L}P_jx_k\|_\infty \\
&+ \|P_if_j\|_\infty + \|RK_jP_j^{-L}P_jx_k\|_\infty - \|P_jx_k\|_\infty \geq \\
&\geq \|P_i(A_j + B_jK_j)x_k + P_if_j\|_\infty + \|RK_jx_k\|_\infty + \|Qx_k\|_\infty - \|P_jx_k\|_\infty.
\end{aligned} \tag{56}$$

Hence, inequality (44) holds.

E. Proof of Lemma VIII.2

We will use induction to prove Lemma VIII.2. For $i = 0$, the inequality $\|x^*_{k+i}\|_p \leq \alpha_i \|x_k\|_p$ holds for any $\alpha_0 \geq 1$. Suppose $\|x^*_{k+i}\|_p \leq \alpha_i \|x_k\|_p$ holds for some $0 \leq i \leq N-2$. Now we will prove that it holds for $i+1$. We have that

$$\|x^*_{k+i+1}\|_p = \|A_j x^*_{k+i} + B_j u^*_{k+i} + f_j\|_p \quad \text{when} \quad x^*_{k+i} \in \mathcal{X}_f(N) \cap \Omega_j, \quad j \in \mathcal{S}.$$

Since there exists a positive number μ such that $\|x\|_p \geq \mu$ for all $x \in \cup_{j \in \mathcal{S}_1} \Omega_j$ and $f_j = 0$ for $j \in \mathcal{S}_0$, it follows that there exists a positive number θ such that $\|f_j\|_p \leq \theta \|x\|_p$ for all $x \in \mathbb{R}^n$ and all $j \in \mathcal{S}$. Then, by Assumption VIII.1 it follows that

$$\begin{aligned} \|x^*_{k+i+1}\|_p &\leq \|A_j\|_p \|x^*_{k+i}\|_p + \|B_j\|_p \|u^*_{k+i}\|_p + \|f_j\|_p \leq \\ &\leq \max_{j \in \mathcal{S}} (\|A_j\|_p + \beta_i \|B_j\|_p + \theta) \|x^*_{k+i}\|_p. \end{aligned} \tag{57}$$

Hence, by the induction hypothesis it follows that

$$\|x^*_{k+i+1}\|_p \leq \alpha_{i+1} \|x_k\|_p,$$

for $\alpha_{i+1} := \max_{j \in \mathcal{S}} (\|A_j\|_p + \beta_i \|B_j\|_p + \theta) \alpha_i > 0$.

REFERENCES

[1] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, pp. 407–427, 1999.

[2] E. Sontag, "Nonlinear regulation: the piecewise linear approach," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 346–357, 1981.

[3] W. Heemels, B. De Schutter, and A. Bemporad, "Equivalence of hybrid dynamical models," *Automatica*, vol. 37, pp. 1085–1091, 2001.

[4] A. Bemporad, "Efficient conversion of mixed logical dynamical systems into an equivalent piecewise affine form," *IEEE Transactions on Automatic Control*, vol. 49, no. 5, pp. 832–838, 2004.

[5] A. Bemporad, F. Borrelli, and M. Morari, "Optimal controllers for hybrid systems: Stability and piecewise linear explicit form," in *39th IEEE Conference on Decision and Control*, Sydney, Australia, 2000, pp. 1810–1815.

[6] E. Kerrigan and D. Mayne, "Optimal control of constrained, piecewise affine systems with bounded disturbances," in *41st IEEE Conference on Decision and Control*, Las Vegas, Nevada, 2002, pp. 1552–1557.

[7] M. Lazar, W. Heemels, S. Weiland, and A. Bemporad, "Stabilization conditions for model predictive control of constrained PWA systems," in *43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 2004, pp. 4595–4600.

[8] S. Qin and T. Badgwell, "A survey of industrial model predictive control technology," *Control Engineering Practice*, vol. 11, pp. 733–764, 2003.

[9] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[10] H. Freeman, *Discrete-time systems*. John Wiley & Sons, Inc., 1965.

[11] W. Hahn, *Stability of motion*. Springer-Verlag, 1967.

[12] J. LaSalle, "The stability of dynamical systems," in *Regional Conference Series in Applied Mathematics*, SIAM, Ed., no. 25, Philadelphia, 1976.

[13] P. Grieder, "Efficient computation of feedback controllers for constrained systems," Ph.D. dissertation, Swiss Federal Institute of Technology (ETH) Zürich, Switzerland, 2004.

[14] F. Christophersen, M. Baotic, and M. Morari, "Stability analysis of hybrid systems with a linear performance index," in *43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 2004, pp. 4589–4594.

[15] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 555–559, 1998.

[16] D. Mignone, G. Ferrari-Trecate, and M. Morari, "Stability and stabilization of piecewise affine systems: An LMI approach," Automatic Control Laboratory, ETH Zürich, Switzerland, Tech. Rep. AUT00-12, 2000.

[17] F. Blanchini, "Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 39, no. 2, pp. 428–433, 1994.

[18] G. Golub and C. Van Loan, *Matrix computations*. The Johns Hopkins University Press, 1989.

[19] P. Scokaert, J. Rawlings, and E. Meadows, "Discrete-time stability with perturbations: Application to model predictive control," *Automatica*, vol. 33, no. 3, pp. 463–470, 1997.

[20] G. Goodwin, M. Seron, and J. De Dona, *Constrained control and estimation An optimization approach*, ser. Communications and control engineering. Springer, 2005.

[21] M. Lazar, W. Heemels, S. Weiland, and A. Bemporad, "Stabilizing receding horizon control of PWL systems: An LMI approach," in *16th Symposium on Mathematical Theory of Networks and Systems*, Leuven, Belgium. Paper downloadable from http://www.mtns2004.be/database/papersubmission/277.pdf or from http://www.cs.ele.tue.nl/MLazar/MTNSpaper.pdf., 2004.

[22] S. Rakovic, P. Grieder, M. Kvasnica, D. Mayne, and M. Morari, "Computation of invariant sets for piecewise affine discrete time systems subject to bounded disturbances," in *43nd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 2004, pp. 1418–1423.

[23] I. Kolmanovsky and E. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, pp. 317–367, 1998.

[24] A. Alessio and A. Bemporad, "A Recursive Algorithm for DC Programming and Applications in Computational Geometry," Dipartimento di Ingegneria dell'Informazione, Universitá di Siena, Via Roma 56, 53100 Siena, Italy, Tech. Rep., 2004.

[25] A. Bemporad, *Hybrid Toolbox – User's Guide*, Dec. 2003, http://www.dii.unisi.it/hybrid/toolbox.

[26] M. Kvasnica, P. Grieder, M. Baotic, and M. Morari, "Multi Parametric Toolbox (MPT)," in *Hybrid Systems: Computation and Control*, ser. Lecture Notes in Computer Science, Volume 2993. Pennsylvania, Philadelphia, USA: Springer Verlag, Mar. 2004, pp. 448–462, urlhttp://control.ee.ethz.ch/ mpt.

[27] M. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.

[28] J. Daafouz, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach," *IEEE Transactions on Automatic Control*, vol. 47, pp. 1883–1887, 2002.

[29] P. Gutman and M. Cwikel, "An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states," *IEEE Transactions on Automatic Control*, vol. AC-32, no. 3, pp. 251–254, 1987.