Set-membership identification of block-structured nonlinear feedback systems

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Nonlinear feedback systems

\[ \nu_t = \mathcal{N}(w_t) = \sum_{k=1}^{n} \gamma_k w_t^k \text{ with } n: \text{ polynomial degree} \]

\[ w_t = \frac{B(q^{-1})}{A(q^{-1})} x_t \text{ with } \]

\[ A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-na} \]

\[ B(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_{nb} q^{-nb} \]

\[ q^{-1} w_t = w_{t-1} \]

\( \mathcal{N} \): nonlinear static block
\( \mathcal{L} \): linear dynamic subsystem
\( x_t, \nu_y \): not measurable inner signals
\( u_t \): known input signal
\( y_t \): noise-corrupted measurement of \( w_t \)
**Problem formulation**

- **Aim:** compute bounds on the parameters $\gamma^T = [\gamma_1, \gamma_2 \ldots \gamma_n]$ and $\theta^T = [a_1 \ldots a_{na} b_0 \ldots b_{nb}]$

- **Prior assumption on the system:**
  - BIBO stability
  - $na$ and $nb$ are known
  - $n$ is finite and known
  - the steady-state gain of the linear subsystem is not zero
  - a rough upper bound on the settling time of the system is known

- **Prior assumption on the measurement uncertainty:**
  - $\eta_t$ is UBB: $|\eta_t| \leq \Delta \eta_t$
  - $\Delta \eta_t$ is known
Proposed solution

Three-stage procedure:

- **First stage**: computation of **bounds** on the nonlinear block parameters $\gamma$.

- **Second stage**: computation of **bounds** on the inner (unmeasurable) signal $x_t$.

- **Third stage**: computation of **bounds** on the linear block parameters $\theta$. 
Proposed solution: first stage

Bounds on the parameters $\gamma$ of the nonlinear block:

- Stimulate the system with square-wave of $M$ different amplitude and get steady-state measurements
- The feasible parameters set $\mathcal{D}_\gamma$ of the nonlinear block is described as:

$$\mathcal{D}_\gamma = \{ \gamma \in \mathbb{R}^n : (\bar{y}_s - \bar{\eta}_s) + \sum_{k=1}^{n} \gamma_k (\bar{y}_s - \bar{\eta}_s)^k = \bar{u}_s, \quad |\bar{\eta}_s| \leq \Delta \bar{\eta}_s; \quad s = 1, \ldots, M \},$$

- $\mathcal{D}_\gamma$ is the set of all parameters $\gamma$ consistent with the $M$ given measurements, the error bounds and the assumed model structure
- Bounds on parameter $\gamma_k$:

$$\gamma_k^{\text{min}} = \min_{\gamma \in \mathcal{D}_\gamma} \gamma_k \quad \quad \quad \quad \gamma_k^{\text{max}} = \max_{\gamma \in \mathcal{D}_\gamma} \gamma_k$$
Proposed solution: first stage

Computation of $\gamma_k^{\text{min}}$ and $\gamma_k^{\text{max}}$:

$$
\gamma_k^{\text{min}} = \min_{(\gamma, \eta) \in D_{\gamma \eta}} \gamma_k \quad \gamma_k^{\text{max}} = \max_{(\gamma, \eta) \in D_{\gamma \eta}} \gamma_k
$$

Here:

$$
\eta = [\eta_1 \eta_2 \ldots \eta_M]^T,
$$

$$
D_{\gamma \eta} = \{ (\gamma, \eta) \in \mathbb{R}^n \times \mathbb{R}^M : (y_s - \eta_s) + \sum_{k=1}^{n} \gamma_k (y_s - \eta_s)^k = u_s, \quad |\eta_s| \leq \Delta \eta_s; \quad s = 1, \ldots, M \}
$$

$D_{\gamma \eta}$ is a semialgebraic set over $\mathbb{R}^{n+M}$

The above problems are semialgebraic (nonconvex) optimization problems
Proposed solution: first stage

Standard nonlinear optimization tools can not be exploited to compute $\gamma^\text{min}_k$ and $\gamma^\text{max}_k$ since they can trap in local minima.

\[
\text{The true value of } \gamma_k \text{ could not lie in } [\gamma^\text{min}_k, \gamma^\text{max}_k]
\]

Relax original identification problems to convex optimization problems.

\[
\text{Bounds on each parameter } \gamma_k \text{ can be obtained}
\]
Convex relaxation

LMI relaxation for semialgebraic optimization problems:

- **SOS decomposition**
  

- **Theory of moments**
  
  J. B. Lasserre, “Global optimization with polynomials and the problem of moments”, *SIAM J. on Opt. 2001*

-relaxed bounds $\gamma_k^{\min \delta}$ and $\gamma_k^{\max \delta}$ computed solving the following SDP problems:

\[
\gamma_k^{\min \delta} = \min_{x \in \mathcal{D}_x^\delta} f(x) \quad \gamma_k^{\max \delta} = \max_{x \in \mathcal{D}_x^\delta} f(x)
\]

here:

- LMI decision variables $x$
- linear function $f(x)$
- Convex set described by LMI constraints
Tightness and convergence

**Property 1** — $\delta$-relaxed bounds become tighter as $\delta$ increases:

\[
\begin{align*}
\gamma_k^{\min \delta} & \leq \gamma_k^{\min \delta + 1} \leq \gamma_k^{\min} \\
\gamma_k^{\max \delta} & \geq \gamma_k^{\max \delta + 1} \geq \gamma_k^{\max}
\end{align*}
\]

**Property 2** — $\delta$-relaxed bounds converge to the true bounds as $\delta \to \infty$:

\[
\begin{align*}
\lim_{\delta \to \infty} \gamma_k^{\min \delta} &= \gamma_k^{\min} \\
\lim_{\delta \to \infty} \gamma_k^{\max \delta} &= \gamma_k^{\max}
\end{align*}
\]
Computational complexity of the LMI relaxation

In practice, due to an high computational complexity, LMI relaxation techniques can be exploited only for a small set of measurements

A reduction of the complexity of SDP relaxed problems is necessary
Reduced complexity of the relaxed problems

\[ \mathcal{D}_{\gamma \eta} = \{ (\gamma, \eta) \in \mathbb{R}^n \times \mathbb{R}^M : (\bar{y}_s - \bar{\eta}_s) + \sum_{k=1}^{n} \gamma_k (\bar{y}_s - \bar{\eta}_s)^k = \bar{u}_s, \]

\[ |\eta_s| \leq \Delta \eta_s; \quad s = 1, \ldots, M \} \]

Property 3 The variables \( \bar{\eta}_s \) defining \( \mathcal{D}_{\gamma \eta} \) are not correlated with each other.

In constructing moment matrix defining \( \mathcal{D}_x^\delta \) do not consider the correlation between variables \( \bar{\eta}_s \).
Reduced complexity of the relaxed problems

Value of $M$ greater than 400 can be exploited in the identification (for $\delta \leq 4$)

Property 4 — Convergence to tight bounds is preserved
Proposed solution: second stage

Bounds on the inner signal $x_t$:

- $x_t^{\text{min}} = u_t - \nu_t^{\text{max}}$
- $x_t^{\text{max}} = u_t - \nu_t^{\text{min}}$
- $\nu_t^{\text{min}} = \min_{(\gamma, \bar{\eta}) \in \mathcal{D}_{\gamma \bar{\eta}}, |\eta_t| \leq \Delta \eta_t} \sum_{k=1}^{n} \gamma_k (y_t - \eta_t)^k$
- $\nu_t^{\text{max}} = \max_{(\gamma, \bar{\eta}) \in \mathcal{D}_{\gamma \bar{\eta}}, |\eta_t| \leq \Delta \eta_t} \sum_{k=1}^{n} \gamma_k (y_t - \eta_t)^k$

- Stimulate the system with a persistently exciting input signal $u_t$
- Bounds on $\nu_t$ can be computed by means of LMI relaxation
- Structure of the problem can be exploited to reduce the computation complexity
Proposed solution: third stage

Bounds on the linear block parameters $\theta$:

Inner signal $x_t$ described in terms of its central value $x^c_t$ and its perturbation $\delta x_t$:

$$x_t = x^c_t + \delta x_t$$

with:

$$|x_t| \leq \Delta x_t, \quad x^c_t = \frac{x_{tmin} + x_{tmax}}{2}, \quad \Delta x_t = \frac{x_{tmax} - x_{tmin}}{2}$$

Identification of a linear model with noisy output sequence $\{y_t\}$ and uncertain input sequence $\{x_t\}$

Errors-in-variables (EIV) problem with bounded errors
Proposed solution: bounds on $\theta$

Exploiting previous results on EIV problems with bounded errors

Cerone, “Feasible parameter set of linear models with bounded errors in all variables”, *Automatica* 1993

$\downarrow$

Bounds on $\theta_j$ are computed by means of linear programming
Example

Parameters of the simulated system

\( (w_t) = -1.5w_t + 1.2w_t^2 + 0.9w_t^3 \)

\( (q^{-1}) = 1 - 1.5193q^{-1} + 0.5326q^{-2} \)

\( (q^{-1}) = 0.1549q^{-1} - 0.1416q^{-2} \)

Measurements output errors

\( |\Delta \bar{\eta}_s|, \{\bar{\eta}_s\} \) random variables belonging to \([-\Delta \bar{\eta}_s, +\Delta \bar{\eta}_s]\)

\( |\Delta \eta_t|, \{\eta_t\} \) random variables belonging to \([-\Delta \eta_t, +\Delta \eta_t]\)

During the simulated experiment the SNR is about \(25db\).
Nonlinear block parameters: central estimates and parameters bounds ($M = 50$, $\delta = 3$)

<table>
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<tr>
<th>True Value</th>
<th>$\gamma_k^{\text{min}}$</th>
<th>$\gamma_k^{c}$</th>
<th>$\gamma_k^{\text{max}}$</th>
<th>$\Delta \gamma_k$</th>
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$$\Delta \gamma_k = \frac{\gamma_k^{\text{max}} - \gamma_k^{\text{min}}}{2}$$
### Linear block parameters: central estimates and parameters bounds

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<tr>
<th>N</th>
<th>True Value</th>
<th>$\theta_j^{\text{min}}$</th>
<th>$\theta_j^c$</th>
<th>$\theta_j^{\text{max}}$</th>
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</table>

\[
\Delta \theta_j = \frac{\theta_j^{\text{max}} - \theta_j^{\text{min}}}{2}
\]
Conclusion

- Three stage procedure to evaluate parameters bounds of a nonlinear feedback system
- Bounds on the nonlinear block parameters have been evaluated by means of LMI relaxation techniques
- The particular structure of the identification problems allows the reduction of the complexity of the LMI relaxation
- Convergence to tight bounds is guaranteed
- Bounds on the parameters of the linear block has been computed through the evaluation of bounds on the unmeasurable inner signal $x_t$