Convex relaxation techniques for set-membership identification of LPV systems

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***Abstract*— Set-membership identification of single-input single-output linear parameter varying models is considered in the paper under the assumption that both the output and the scheduling parameter measurements are affected by bounded noise. First, we show that the problem of computing the parameter uncertainty intervals requires the solutions to a number of nonconvex optimization problems. Then, on the basis of the analysis of the regressor structure, we present some *ad hoc* convex relaxation schemes to compute parameter bounds by means of semidefinite optimization. Advantages of the new techniques with respect to previously published results are discussed both theoretically and by means of simulations.**

***Index Terms*— Bounded error identification, Linear Parameter Varying, LMI relaxation, Parameters bounds.**

I. Introduction

Linear parameter varying (LPV) models belong to the more general class of linear time-varying models and, roughly speaking, they can be defined as linear systems where either the matrices of the state equations or the coefficients of the difference equation relating the input and the output signals depend on one or more time varying parameters whose real-time measurements are assumed to be available. These models have received a considerable attention from the identification and control community in recent years and can now be considered as one of the most popular tool to derive mathematical description of nonlinear/time-varying phenomena. As to the identification of LPV models, a relevant number of approaches has appeared in the literature since the work by Nemani, Ravikanth and Bamieh [1] which seems to be the first paper addressing the problem. They consider linear parameter varying models with a single time-varying parameter and assume that the measurements of all the state variables are available. A parameter estimation scheme based on the minimization of a prediction error cost function is proposed in [2] where LPV models with multiple time-varying parameters are considered under the assumption of LFT parameter dependence. Least mean square and recursive least square algorithms are proposed in [3] to solve the identification of LPV input-output models assuming that measurements of input, output and scheduling parameters are available. Persistency of excitation conditions in terms of inputs and scheduling parameters trajectories are also derived. Subspace identification of multiple input multiple output (MIMO) LPV models with affine parameters dependence is considered in a number of papers. In [4] it is shown that standard subspace algorithms cannot be used in practice to identify LPV models since the dimensions of the data matrices grow exponentially with the system order. Significant improvements over the method proposed in [4] are presented in [5], exploiting kernel methods, and in [6] where an instrumental variable approach is considered and the positive effect of using periodic scheduling sequences is highlighted. Iterative subspace system identification is considered in [7]. Application of LPV subspace identification algorithms to both periodic and nonlinear systems are discussed in [8] where the proposed approach is applied to the modeling of helicopter rotor dynamics. Separable least squares are exploited in [9] to derive a novel algorithm for a class of nonlinear parameter-varying models represented in the form of a linear fractional transformation, while an orthonormal basis functions based approach is presented in [10]. A detailed overview of the available LPV modeling and identification approaches can be found in the recent book [11] by Toth, where the nontrivial relation between state-space and input-output description for LPV systems is also discussed. In all the papers mentioned above, the measurement error is statistically described. An alternative to the stochastic description of measurement errors is the bounded-errors or set-membership characterization, where uncertainties are assumed to belong to a given set (see, e.g., [12]). In this context, all parameters belonging to the *feasible parameter set* (*FPS*), i.e. parameters consistent with measurements, error bounds and the assumed model structure, are feasible solutions to the identification problem. To the authors best knowledge, only a couple of contributions address the identification of LPV models when measurement errors are supposed to be bounded. In [13] the problem of identification and model validation of LPV systems in the presence of bounded noise and a possible nonparametric part is considered. A solution is proposed recasting the problem in terms of checking the feasibility of a set of linear matrix inequalities. In [14] the author consider the identification of discrete-time LPV models with finite impulse response structure and output measurements affected by bounded noise.

In this paper a procedure for set-membership identification of SISO discrete-time LPV models when both the output and the time-varying parameters measurements are affected by bounded noise is considered. Preliminary results on this problem are presented in [15] and successfully applied to the problem of deriving an LPV model of the vehicle

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lateral dynamics in [16]. Thanks to a careful analysis of the problem structure, a new convex relaxation approach is proposed in this paper to compute the parameter uncertainty intervals by means of semidefinite optimization. The obtained bounds are proven to be tighter than those obtained in [15]. The paper is organized as follows. Section II is devoted to the problem formulation. In Section III we show that computation of tight parameters bounds requires the solution to nonconvex optimization problems. A brief review of the algorithm proposed in [15] is presented in Section IV, while the new identification procedure is described in Section V. A simulated example is reported in Section VI in order to highlight the improvement of the presented procedure in the evaluation of the uncertainty intervals with respect to the algorithm proposed in [15].

II. Problem formulation

Consider the SISO discrete-time LPV model described in terms of the linear difference equation

$$\mathcal{A}(q^{-1},\lambda_t)w_t = \mathcal{B}(q^{-1},\lambda_t)u_t, \tag{1}$$

where u_t and w_t are the input and the output signals respectively, while $\lambda_t \in \mathbb{R}^{\mu}$, $\lambda_t = [\lambda_{1_t}\lambda_{2_t}\ldots\lambda_{\mu_t}]^{\mathrm{T}}$ is a vector of time-varying parameters which, according to the LPV modeling and control literature (see, e.g., [17]) are assumed to be measurable. $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$ are polynomials in the backward shift operator q^{-1},

$$\mathcal{A}(q^{-1},\lambda_t) = 1 + a_1(\lambda_t)q^{-1} + \ldots + a_{na}(\lambda_t)q^{-na}, \tag{2}$$

$$\mathcal{B}(q^{-1},\lambda_t) = b_0(\lambda_t) + b_1(\lambda_t)q^{-1} + \ldots + b_{nb}(\lambda_t)q^{-nb} \tag{3}$$

where $na \geq nb$ and the coefficients a_i and b_j are assumed to be nonlinear memoryless mappings of parameters λ_t described by

$$a_i(\lambda_t) = \sum_{k=1}^{n_i} a_{i,k}\phi_{i,k}(\lambda_t), \tag{4}$$

$$b_j(\lambda_t) = \sum_{h=0}^{m_j} b_{j,h}\psi_{j,h}(\lambda_t), \tag{5}$$

where $\phi_{i,k}(\cdot)$ and $\psi_{j,h}(\cdot)$ are known nonlinear basis functions. In our work we assume that $\phi_{i,k}(\cdot)$ and $\psi_{j,h}(\cdot)$ belong to the canonical polynomial basis in the parameters λ_t, and we denote as $d_{\phi_{i,k}}$ and $d_{\psi_{j,h}}$ the degree of $\phi_{i,k}(\cdot)$ and $\psi_{j,h}(\cdot)$, respectively. Let y_t and z_t be the noise-corrupted measurements of w_t and λ_t respectively

$$y_t = w_t + \eta_t, \tag{6}$$

$$z_t = \lambda_t + \varepsilon_t, \tag{7}$$

where $\varepsilon_t = [\varepsilon_{1_t}\varepsilon_{2_t}\ldots\varepsilon_{\mu_t}]^{\mathrm{T}}$. Measurements uncertainties η_t and ε_{s_t} are known to range within given bounds $\Delta\eta_t$ and $\Delta\varepsilon_{s_t}$, more precisely

$$|\eta_t| \leq \Delta\eta_t, \tag{8}$$

and

$$\varepsilon_t \in \mathcal{E} = \{\varepsilon_t \in \mathbb{R}^{\mu} : |\varepsilon_{s_t}| \leq \Delta\varepsilon_{s_t}, s = 1,2,\ldots,\mu\} \tag{9}$$

The unknown parameter vector $\theta \in \mathbb{R}^{n_\theta}$ to be estimated is defined as

$$\begin{aligned}\theta^{\mathrm{T}} = [&a_{1,1}\ldots a_{1,n_1}\ldots a_{na,1}\ldots a_{na,n_{na}}\\ &b_{0,0}\ldots b_{0,m_1}\ldots b_{nb,1}\ldots b_{nb,m_{nb}}],\end{aligned} \tag{10}$$

where $n_\theta = \sum_{i=1}^{na} n_i + \sum_{j=0}^{nb} m_j$. In this paper we address the problem of deriving uncertainty intervals on the parameters θ. For the sake of simplicity and without loss of generality, in the rest of the paper we only consider the case of a scalar scheduling variable λ_t, that is $\lambda_t \in \mathbb{R}$.

III. Evaluation of tight parameters bounds

The set $\mathcal{D}$ of all the LPV system parameters θ and the noise samples ξ_t and η_t consistent with the measurement data sequence, the assumed model structure and the error bounds is described by equations (1) - (9), i.e.

$$\begin{aligned}\mathcal{D} = \Big\{&(\theta,\eta,\varepsilon) \in \mathbb{R}^{n_\theta+N+(N-na)} :\\ &\mathcal{A}(q^{-1},z_t-\varepsilon_t)[y_t-\eta_t] = \mathcal{B}(q^{-1},z_t-\varepsilon_t)u_t,\\ &|\varepsilon_t| \leq \Delta\varepsilon_t,\ \ |\eta_r| \leq \Delta\eta_r,\\ &t = na+1,\ldots,N;\ r = 1,\ldots,N\Big\}.\end{aligned} \tag{11}$$

with $\eta = [\eta_1,\ldots,\eta_N]^{\mathrm{T}}$ and $\varepsilon = [\varepsilon_{na+1},\ldots,\varepsilon_N]^{\mathrm{T}}$. Therefore, for $j = 1,\ldots,n_\theta$, tight bounds on the parameter θ_j can be computed by solving the optimization problems

$$\underline{\theta}_j = \min_{(\theta,\eta,\varepsilon)\in\mathcal{D}} \theta_j, \quad \overline{\theta}_j = \max_{(\theta,\eta,\varepsilon)\in\mathcal{D}} \theta_j. \tag{12}$$

Parameter uncertainty intervals on θ_j are defined as $PUI_j = [\underline{\theta}_j;\ \overline{\theta}_j]$. Because of the polynomial constraints $\mathcal{A}(q^{-1},z_t-\varepsilon_t)[y_t-\eta_t] = \mathcal{B}(q^{-1},z_t-\varepsilon_t)u_t$ defining the feasible region $\mathcal{D}$, problems (12) are nonconvex. Therefore, standard nonlinear optimization tools (gradient method, Newton method, etc.) can not be used because they can trap in local minima/maxima. As a consequence, the PUI_j obtained using these tools is not guaranteed to contain the true unknown parameter θ_j, which is a key requirement of any set-membership identification method. One possible solution to overcome such a problem is to relax the identification problems (12) to convex optimization problems in order to numerically compute lower bounds of $\underline{\theta}_j$ as well as upper bounds of $\overline{\theta}_j$. It can be shown (see [18]) that (12) are semialgebraic optimization problems with an inherent structured sparsity. Then, approximate solutions of $\underline{\theta}_j$ and $\overline{\theta}_j$ can be computed through a direct implementation of the *sparse* LMI relaxation techniques described in [19] and [20]. Unfortunately, due to high memory usage, the relaxation order δ has to be rather low to implement such an identification procedure in a commercial workstation. Roughly speaking, δ should be not greater than 2 when the number of parameters θ is about 6 and the number N of measurements is about 30. In order to deal with problems with a larger number of measurements and parameters, a relaxation method, called static

LPV relaxation, for evaluating parameter bounds of LPV systems in the set-membership context is proposed in [15] and it is briefly reviewed in Section IV for self-consistency of the paper. In this work we propose an alternative method, called partial-dynamic LPV relaxation, which reduces the computational complexity of identification problems (12), so that the *sparse* LMI relaxation techniques described in [19] and [20] can be used to compute guaranteed parameter bounds. Such a method provides parameter bounds tighter than the ones obtained in [15].

IV. Static LPV relaxation

To the authors best knowledge, only one algorithm is available in the literature to evaluate parameter bounds for LPV systems when both the measurements on the output and on the time-varying parameters are affected by bounded noise. Such a method, called static LPV relaxation, was proposed in [15] where an outer-approximation $\mathcal{D}^s$ of the feasible set $\mathcal{D}$ has been constructed. In particular, the set $\mathcal{D}^s$ is defined by piecewise linear constraints and, although generally nonconvex, it is the union of at most 2^{n_θ} polytopes in the parameters space $\mathbb{R}^{n_\theta}$. Relaxed parameter bounds $\underline{\theta}_j^s$ and $\overline{\theta}_j^s$ are computed by solving the optimization problems

$$\underline{\theta}_j^s = \min_{\theta \in \mathcal{D}_\theta^s} \theta_j, \quad \overline{\theta}_j^s = \max_{\theta \in \mathcal{D}_\theta^s} \theta_j. \tag{13}$$

The relaxed parameter uncertainty interval PUI_j^s, defined as $PUI_j^s = \left[\underline{\theta}_j^s,\ \overline{\theta}_j^s\right]$, is guaranteed to contain the true unknown parameter θ_j, that is $\theta_j \in PUI_j^s$, for every $j = 1, \ldots, n_\theta$.

V. Partial-dynamic LPV relaxation

In this section we present a new technique to relax (12) to convex optimization problems. For the sake of clarity, a general overview of the proposed method is first presented in Section V-A. Then, detailed technical results are provided in Section V-B.

A. Overview of the relaxation procedure

Let us rewrite $\mathcal{D}$, defined by (11), in the matrix form

$$\begin{aligned}\mathcal{D} = \Big\{&(\theta,\eta,\varepsilon) \in \mathbb{R}^{n_\theta+2N-na} : \\ &A\left[\theta^{\mathrm{T}} 1\right]^{\mathrm{T}} = 0,\ \ |\varepsilon_t| \le \Delta\varepsilon_t,\ \ |\eta_r| \le \Delta\eta_r \\ &t = na+1,\ldots,N;\ r = 1,\ldots,N\Big\}.\end{aligned} \tag{14}$$

where $A \in \mathbb{R}^{N-na, n_\theta+1}$ and the t-th row A_t of A is

$$\begin{aligned}A_t = [&(y_{t+na} - \eta_{t+na})\phi_{1,1}(z_{t+na+1} - \varepsilon_{t+na+1}), \ldots, \\ &(y_t - \eta_t)\phi_{na,n_{na}}(z_{t+na+1} - \varepsilon_{t+na+1}), \\ &u_{t+na+1}\psi_{0,0}(z_{t+na+1} - \varepsilon_{t+na+1}), \ldots, \\ &u_{t+na-nb}\psi_{nb,m_{nb}}(z_{t+na+1} - \varepsilon_{t+na+1}), \\ &-y_{t+na+1} + \eta_{t+na+1}] \, .\end{aligned} \tag{15}$$

Note that the rows of the matrix A are correlated with each other since the noise variables η_t appears in all the rows A_i, with $i = t-na, t-na+1, \ldots, t$. Besides, also the columns of the matrix A are not independent of each other, since they are correlated by the noise variable affecting the scheduling parameter. The main idea of the partial-dynamic LPV relaxation can be summarized in the following steps:

(i) First, consider the rows of the matrix A independent with each other, keeping the correlation between the columns. This leads to the construction of an outer-bound $\mathcal{D}^r$ of the original feasible set $\mathcal{D}$.

(ii) Then, consider the columns of the matrix A independent with each other, keeping the correlation between the rows. This leads to the construction of another outer-bound $\mathcal{D}^c$ of $\mathcal{D}$.

(iii) Define the relaxed feasible parameter set $\mathcal{D}^{rc} = \mathcal{D}^r \cap \mathcal{D}^c$ and, for every $j = 1, \ldots, n_\theta$, compute minimum and maximum value of the parameters θ_j over the feasible set $\mathcal{D}^{rc}$.

B. Technical results

***Result** 1:* **Construction of the set $\mathcal{D}^r$**
Let us define the set $\mathcal{D}^r$ as

$$\begin{aligned}\mathcal{D}^r = \Big\{&(\theta,\eta,\varepsilon) \in \mathbb{R}^{n_\theta+2N-na} : \\ &\overline{A}_t^r\theta \ge y_t - \Delta\eta_t,\ \ \underline{A}_t^r\theta \le y_t + \Delta\eta_t, \\ &|\varepsilon_t| \le \Delta\varepsilon_t,\ \ t = na+1,\ldots,N\Big\}\,,\end{aligned} \tag{16}$$

where $\overline{A}_t^r$ and $\underline{A}_t^r$ are

$$\begin{aligned}\overline{A}_t^r = \big[&(-y_{t-1} + \Delta\eta_{t-1} sgn\big(\phi_{1,1}(z_t - \varepsilon_t)\big) sgn(a_{1,1}))\phi_{1,1}, \ldots, \\ &(-y_{t-na} + \Delta\eta_t sgn\big(\phi_{na,n_{na}}(z_t - \varepsilon_t)\big) sgn(a_{na,n_{na}}))\phi_{na,n_{na}}, \\ &u_t\psi_{0,0}(z_t - \varepsilon_t), \ldots, u_{t-nb}\psi_{nb,m_{nb}}(z_t - \varepsilon_t)\big]\,.\end{aligned} \tag{17}$$

and

$$\begin{aligned}\underline{A}_t^r = \big[&(-y_{t-1} - \Delta\eta_{t-1} sgn\big(\phi_{1,1}(z_t - \varepsilon_t)\big) sgn(a_{1,1}))\phi_{1,1}, \ldots, \\ &(-y_{t-na} - \Delta\eta_t sgn\big(\phi_{na,n_{na}}(z_t - \varepsilon_t)\big) sgn(a_{na,n_{na}}))\phi_{na,n_{na}}, \\ &u_t\psi_{0,0}(z_t - \varepsilon_t), \ldots, u_{t-nb}\psi_{nb,m_{nb}}(z_t - \varepsilon_t)\big]\,.\end{aligned} \tag{18}$$

The set $\mathcal{D}^r$ is an outer approximation of $\mathcal{D}$. ■

***Remark** 1:* It can be proven that if the noise samples η_t appearing in the rows of the matrix A defined in (17) are not correlated, then $\mathcal{D} = \mathcal{D}^r$.

In order to construct the outer-approximation $\mathcal{D}^c$ of $\mathcal{D}$ we first provide the following definitions:

$$\underline{\phi}_{i,k}^t = \min_{|\varepsilon_t| \le \Delta\varepsilon_t} \phi_{i,k}^t(z_t - \varepsilon_t), \quad \overline{\phi}_{i,k}^t = \max_{|\varepsilon_t| \le \Delta\varepsilon_t} \phi_{i,k}^t(z_t - \varepsilon_t), \tag{19}$$

$$\underline{\gamma}_{j,h}^t = \min_{|\varepsilon_t| \le \Delta\varepsilon_t} \gamma_{j,h}^t, \quad \overline{\gamma}_{j,h}^t = \max_{|\varepsilon_t| \le \Delta\varepsilon_t} \gamma_{j,h}^t, \tag{20}$$

and

$$c(\phi_{i,k}^t) = \frac{\overline{\phi}_{i,k}^t + \underline{\phi}_{i,k}^t}{2},\ \Delta\phi_{i,k}^t = \frac{\overline{\phi}_{i,k}^t - \underline{\phi}_{i,k}^t}{2}, \tag{21}$$

$$c(\gamma_{j,h}^t) = \frac{\overline{\gamma}_{j,h}^t + \underline{\gamma}_{j,h}^t}{2},\ \Delta\gamma_{j,h}^t = \frac{\overline{\gamma}_{j,h}^t - \underline{\gamma}_{j,h}^t}{2} \tag{22}$$

Result 2: **Construction of the set $\mathcal{D}^c$**
Let us define the set $\mathcal{D}^c$ as

$$\mathcal{D}^c = \Big\{(\theta,\eta,\varepsilon) \in \mathbb{R}^{n_\theta+2N-na} : \\ (A_t^c + \Delta A_t^c)\theta \geq y_t - \eta_t, \;\; (A_t^c - \Delta A_t^c)\theta \leq y_t - \eta_t, \\ |\eta_t| \leq \Delta\eta_t, \;\; t = na+1,\ldots,N, \Big\}, \tag{23}$$

where

$$A_t^c = \big[-(y_{t-1}-\eta_{t-1})c(\phi_{1,1}^t),\ldots,-(y_{t-na}-\eta_{t-na})c(\phi_{na,n_{na}}^t), \\ c(\gamma_{0,0}^t),\ldots,c(\gamma_{nb,m_{nb}}^t)\big], \tag{24}$$

$$\Delta A_t^c = \big[(y_{t-1}-\eta_{t-1})\Delta\phi_{1,1}^t sgn(y_{t-1}-\eta_{t-1})sgn(a_{1,1}),\ldots \\ (y_{t-na}-\eta_{t-na})\Delta\phi_{na,n_{na}}^t sgn(y_{t-na}-\eta_{t-na})sgn(a_{na,n_{na}}), \\ \Delta\gamma_{0,0}^t sgn(b_{0,0}),\ldots,\Delta\gamma_{nb,m_{nb}}^t sgn(b_{nb,m_{nb}})\big], \tag{25}$$

Then, the set $\mathcal{D}^c$ is an outer approximation of $\mathcal{D}$. ∎

Proofs of Results 1 and 2 can be found in [18].

Remark 2: It can be proven that if the noise samples λ_t appearing in the elements of the row A_t in (15) are not correlated, then $\mathcal{D} = \mathcal{D}^c$.

Remark 3: Since $\phi_{i,k}^t(\cdot)$ and $\psi_{j,h}^t(\cdot)$ are continuous functions, the Weierstrass theorem guarantees that $\phi_{i,k}^t(\cdot)$ and $\gamma_{j,h}^t(\cdot)$ achieve their global minimum and maximum on the closed interval $|\varepsilon_t| \leq \Delta\varepsilon_t$. As is well known, such a global minimum and maximum must either be stationary points or lie on the boundary of the interval $|\varepsilon_t| \leq \Delta\varepsilon_t$ and their computation is straightforward as $\phi_{i,k}^t(\cdot)$ and $\psi_{j,h}^t(\cdot)$ are polynomial functions.

An outer-approximation of $\mathcal{D}$ tighter than both $\mathcal{D}^r$ and $\mathcal{D}^c$ can be defined as the intersection of $\mathcal{D}^r$ and $\mathcal{D}^c$, i.e. $\mathcal{D}^{rc} = \mathcal{D}^r \cap \mathcal{D}^c$. Then, bounds on the parameters θ_j can be computed by solving the optimization problems

$$\underline{\theta}_j^{pd} = \min_{(\theta,\eta,\varepsilon)\in\mathcal{D}^{rc}} \theta_j, \quad \overline{\theta}_j^{pd} = \max_{(\theta,\eta,\varepsilon)\in\mathcal{D}^{rc}} \theta_j, \tag{26}$$

and parameter uncertainty intervals on θ_j obtained through the partial-dynamic LPV relaxation are then defined as $PUI_j^{pd} = \left[\underline{\theta}_j^{pd};\, \overline{\theta}_j^{pd}\right]$.

Property 1: **Accuracy improvement of PUI_j^{pd} over PUI_j^s**
For every $j = 1,\ldots,n_\theta$, the parameter uncertainty interval PUI_j^{pd} is tighter than the interval PUI_j^s (obtained through the static LPV relaxation [15]), i.e. $PUI_j^{pd} \subseteq PUI_j^s$. ∎

Proof of Property 1 is based on the fact the set $\mathcal{D}^s$ is an outer approximation of both $\mathcal{D}^r$ and $\mathcal{D}^c$. Technical details can be found in [18].

By exploiting the particular structure of the set $\mathcal{D}^{rc}$, we now show as parameter bounds $\underline{\theta}_j^{pd}$ and $\overline{\theta}_j^{pd}$ can be computed through the solution of a set of semialgebraic optimization problems.

Property 2: **Topological features of the feasible set $\mathcal{D}^{rc}$**
If the relative measurement error on the output w_t and on the scheduling variable λ_t is smaller than 100%, then the set $\mathcal{D}^{rc}$ is the union of at most 2^{n_θ} sets $\mathcal{D}_i^{rc}$ in $\mathbb{R}^{n_\theta+2N-na}$, i.e.

$$\mathcal{D}^{rc} = \bigcup_{i=1}^{2^{n_\theta}} \mathcal{D}_i^{rc}. \tag{27}$$

The set $\mathcal{D}_i^{rc}$ is the intersection of $\mathcal{D}^{rc}$ with the i-th orthant $\mathcal{O}_i$ of the parameters space $\mathbb{R}^{n_\theta}$, i.e.

$$\mathcal{D}_i^{rc} = \mathcal{D}^{rc} \cap \mathcal{O}_i, \tag{28}$$

The orthant $\mathcal{O}_i$ is formally described as

$$\mathcal{O}_i = \{\theta \in \mathbb{R}^{n_\theta} : \alpha_{ij}\theta_j \geq 0, j = 1,\ldots,n_\theta\}, \tag{29}$$

where $\alpha_i \in \Gamma$, being Γ the set of all n_θ-dimensional vectors with components equal to ± 1.
Each set $\mathcal{D}_i^{rc}$, if not empty, is a semialgebraic region in $\mathbb{R}^{n_\theta+2N-na}$ defined by polynomial inequalities of maximum degree $d_\theta^{rc} = \max\{1+\max\limits_{i,k}\{d_{\phi_{i,k}}\}, 1+\max\limits_{j,h}\{d_{\psi_{j,h}}\}, 2\}$. ∎

Proof of Property 2 follows since in the orthant $\mathcal{O}_i$ the sign of the parameters θ, appearing in the definition of both $\mathcal{D}_i^r$ and $\mathcal{D}_i^c$, is known. Besides, when the relative measurement error on both the output w_t and on the scheduling variable λ_t is smaller than 100%, also the sign of $y_t - \eta_t$ and $z_t - \varepsilon_t$, appearing the definition of $\mathcal{D}^c$ and $\mathcal{D}^r$ respectively, is known. See [18] for technical details.

Remark 4: The assumption, reasonable in practice, that the relative error on the measurements of w_t and λ_t is smaller than 100% implies that the sign of $y_t - \eta_t$ and $z_t - \varepsilon_t$ is known. If such an assumption is not satisfied, then the set $\mathcal{D}^{rc}$ is the union of at most $2^{n_\theta+2N-na}$ semialgebraic sets.

Thanks to Property 2, identification problems (26) can be decomposed into a collection of polynomial optimization problems. In fact, solving (26) is equivalent to compute

$$\underline{\theta}_j^{pd} = \min_{l=1,\ldots,2^{n_\theta}} \underline{\theta}_{ji}^{pd}; \qquad \overline{\theta}_j^{pd} = \max_{l=1,\ldots,2^{n_\theta}} \overline{\theta}_{ji}^{pd}, \tag{30}$$

where $\underline{\theta}_j^{pd}$ and $\overline{\theta}_j^{pd}$ are the solutions to the following the semialgebraic optimization problems:

$$\underline{\theta}_{ji}^{pd} = \min_{(\theta,\eta,\varepsilon)\in\mathcal{D}_i^{rc}} \theta_j; \qquad \overline{\theta}_{ji}^{pd} = \max_{(\theta,\eta,\varepsilon)\in\mathcal{D}_i^{rc}} \theta_j. \tag{31}$$

The inherent structured sparsity of problems (31), which will be highlighted by Property 3, is exploited to formulate sparse LMI-relaxed problems for (31) in order to compute lower (respectively upper) bounds of $\underline{\theta}_j^{pd}$ (respectively $\overline{\theta}_j^{pd}$).

By rewriting the feasible region $\mathcal{D}_i^{rc}$ defined by (28) as

$$\begin{aligned}\mathcal{D}_i^{rc} = \Big\{ & (\theta,\eta,\varepsilon) \in \mathbb{R}^{n_\theta+2N-na}: \\ & g_t(\theta,\eta,\varepsilon) = \overline{A}_t^r\theta - y_t + \Delta\eta_t \geq 0, \\ & g_{t+N}(\theta,\eta,\varepsilon) = -\underline{A}_t^r\theta + y_t + \Delta\eta_t \geq 0, \\ & g_{t+2N}(\theta,\eta,\varepsilon) = (A_t^c + \Delta A_t^c)\theta - y_t + \eta_t \geq 0, \\ & g_{t+3N}(\theta,\eta,\varepsilon) = -(A_t^c - \Delta A_t^c)\theta + y_t - \eta_t \geq 0, \\ & g_{t+4N}(\theta,\eta,\varepsilon) = \Delta\varepsilon_t - \varepsilon_t \geq 0, \\ & g_{t+5N}(\theta,\eta,\varepsilon) = \Delta\varepsilon_t + \varepsilon_t \geq 0, \\ & g_{r+6N}(\theta,\eta,\varepsilon) = \Delta\eta_r - \eta_r \geq 0, \\ & g_{r+7N}(\theta,\eta,\varepsilon) = \Delta\eta_r + \eta_r \geq 0, \\ & g_{j+8N}(\theta,\eta,\varepsilon) = \alpha_{ij}\theta_j \geq 0, \;\; \alpha_i \in \Gamma \\ & t = na+1,\ldots,N;\; r = 1,\ldots,N;\; j = 1,\ldots,n_\theta \Big\}, \end{aligned} \tag{32}$$

the inherent structured sparsity of problems (31) can be easily detected as described by the following property.

***Property** 3:* Problems (31) enjoy the following features:

P 3.1: The functional involves only the variable θ_j.

P 3.2: For every $t = na+1,\ldots,N$, constraints $g_t \geq 0$ and $g_{t+N} \geq 0$, defining $\mathcal{D}_i^{rc}$ in (32), depend only on the parameters θ and the noise sample ε_t.

P 3.3: For every $t = na+1,\ldots,N$, constraints $g_{t+2N} \geq 0$ and $g_{t+3N} \geq 0$ depend only on the parameters θ and the noise samples η_{t-i} (for $i = 0,1,\ldots,na$).

P 3.4: For every $t = na+1,\ldots,N$, constraints $g_{t+4N} \geq 0$ and $g_{t+5N} \geq 0$ depend only on the noise variable ε_t.

P 3.5: For every $r = 1,\ldots,N$, constraints $g_{r+6N} \geq 0$ and $g_{r+7N} \geq 0$ depend only on the noise variable η_r.

P 3.6: For every $j = 1,\ldots,n_\theta$, the constraint $g_{j+8N} \geq 0$ depends only on the variable θ_j. ■

Thanks to the structured sparsity of problems (31) highlighted by Property 3, the SDP relaxation proposed in [19] and [20] can be applied to problems (31), leading to approximate solutions $\underline{\theta}_{ji}^{pd,\delta}$ and $\overline{\theta}_{ji}^{pd,\delta}$ that are computed by solving the convex SDP problems

$$\underline{\theta}_{ji}^{pd,\delta} = \min_{p \in \mathcal{D}_i^{pd,\delta}} f_j(p), \;\; \overline{\theta}_{ji}^{pd,\delta} = \max_{p \in \mathcal{D}_i^{pd,\delta}} f_j(p), \tag{33}$$

where δ is a given relaxation order, p is the decision variable vector, the objective function $f_j(p)$ is linear in the variables p and the feasible region $\mathcal{D}_i^{pd,\delta}$ is a convex set defined by linear matrix inequalities (LMIs), which takes into account the polynomial constraints defining the semialgebraic set $\mathcal{D}_i^{rc}$ of problems (31). In particular, the number of optimization variables p is $O(N(n_\theta+na)^{2\delta})$, while the size of the LMI describing $\mathcal{D}_i^{pd,\delta}$ is $O(N(n_\theta+na)^{\delta})$. See [18] for technical details on the computation of the number of optimization variables p and of the dimension of the LMI describing the feasible region $\mathcal{D}^{pd,\delta}$.

The minimum value $\underline{\delta}$ of the LMI relaxation order, so that (33) are well-defined, is $\lceil \frac{\rho(\mathcal{D}_i^{rc})}{2} \rceil$, where $\lceil\cdot\rceil$ is the ceiling operator and $\rho(\mathcal{D}_i^{pd,\delta})$ denotes the maximum order of the polynomial constraints defining $\mathcal{D}_i^{rc}$. From Property 2 the maximum degree of the polynomial constraints describing $\mathcal{D}_i^{pd,\delta}$ is equal to d_θ^{rc}, therefore $\underline{\delta} = \lceil \frac{d_\theta^{rc}}{2} \rceil$. The reader is referred to [19] for details on the relaxation of sparse polynomial optimization problems through LMI-based relaxation techniques.

For a given relaxation order $\delta \geq \underline{\delta}$, let us define the δ-relaxed uncertainty intervals obtained through the partial-dynamic-LPV procedure as $PUI_j^{pd,\delta} = \left[\underline{\theta}_j^{pd,\delta};\, \overline{\theta}_j^{pd,\delta}\right]$, where

$$\underline{\theta}_j^{pd,\delta} = \min_{i=1,\ldots,2^{n_\theta}} \underline{\theta}_{ji}^{pd,\delta}, \quad \overline{\theta}_j^{pd,\delta} = \max_{i=1,\ldots,2^{n_\theta}} \overline{\theta}_{ji}^{pd,\delta}. \tag{34}$$

***Property** 4:* For every $j = 1,\ldots,n_\theta$, the δ-relaxed parameter uncertainty interval $PUI_j^{pd,\delta}$ satisfies the following properties.

P 4.1: **Guaranteed relaxed uncertainty intervals.**
For any relaxation order $\delta \geq \underline{\delta}$, the δ-relaxed parameter uncertainty interval $PUI_j^{pd,\delta}$ is guaranteed to contain the true unknown parameter θ_j to be estimated, i.e. $\theta_j \in PUI_j^{pd,\delta}$.

P 4.2: **Monotone convergence to intervals PUI_j^{pd}.**
The δ-relaxed parameter uncertainty interval $PUI_j^{pd,\delta}$ becomes tighter as the relaxation order δ increases, that is

$$PUI_j^{pd,\delta+1} \subseteq PUI_j^{pd,\delta}. \tag{35}$$

Further, as the LMI relaxation order goes to infinity, the δ-relaxed parameter uncertainty interval $PUI_j^{pd,\delta}$ converges to the interval PUI_j^{pd}. ■

The proof of Properties P4.1 and P4.2 (see [18] for details) follows from properties of monotone converge of sparse LMI-relaxation techniques.

VI. Simulated Example

In this section we propose a simulated example in order to show the effectiveness of the presented identification procedure and the accuracy improvement on the parameter bounds evaluation with respect to the static LPV relaxation. The considered LPV system is described by (1) with $\mathcal{A}(q^{-1},\lambda_t) = 1 + 0.7\lambda_t q^{-1} + (-0.4 + 0.3\lambda_t^2)q^{-2}$ and $\mathcal{B}(q^{-1},\lambda_t) = 0.1q^{-1} + (1.1\lambda_t + 0.3\lambda_t^2)q^{-2}$. Therefore, the true parameters vector is $\theta = [a_{1,1},\ a_{2,1},\ a_{2,2},\ b_{1,1},\ b_{2,1},\ b_{2,2}]^{\mathrm{T}} = [0.7,\ -0.4,\ 0.3,\ 0.1,\ 1.1,\ 0.3]^{\mathrm{T}}$ and the functions $\phi_{i,k}$ and $\psi_{j,h}$ in (4) and (5), which depend on the scheduling parameter λ_t, are $\phi_{1,1} = \lambda_t$, $\phi_{2,1} = 1$, $\phi_{2,2} = \lambda_t^2$, $\psi_{1,1} = 1$, $\psi_{2,1} = \lambda_t$ and $\psi_{2,2} = \lambda_t^2$. The input sequence $\{u_t\}$ is a random uniform distributed signal which takes values in the interval $[-1,\ 1]$, while $\lambda_t = 2\sin(0.1t)$. The output w_t and the scheduling signal λ_t are corrupted by random additive noises η_t and ε_t, respectively, uniformly distributed in $[-\Delta\eta_t,\ +\Delta\eta_t]$ and $[-\Delta\varepsilon_t,\ +\Delta\varepsilon_t]$. The chosen error bounds $\Delta\eta_t$ and $\Delta\varepsilon_t$ are such that the signal to noise ratios on the output SNR_w and on the scheduling signal SNR_λ, defined as $SNR_w = 10\log\left\{\sum_{t=1}^{N} w_t^2 \Big/ \sum_{t=1}^{N} \eta_t^2\right\}$ and $SNR_\lambda = 10\log\left\{\sum_{t=1}^{N} \lambda_t^2 \Big/ \sum_{t=1}^{N} \varepsilon_t^2\right\}$, are equal to 27

Table I: Parameter central estimates (θ_j^{cs}), parameter bounds ($\underline{\theta}_j^s$, $\overline{\theta}_j^s$) and parameter uncertainties $\Delta\theta_j^s$ obtained through the static LPV relaxation

Parameter	True Value	$\underline{\theta}_j^s$	θ_j^{cs}	$\overline{\theta}_j^s$	$\Delta\theta_j^s$
$a_{1,1}$	0.700	0.546	0.729	0.913	0.1837
$a_{2,1}$	-0.400	-0.472	-0.409	-0.347	0.0627
$a_{2,2}$	0.300	0.196	0.325	0.454	0.1293
$b_{1,1}$	0.100	0.074	0.101	0.128	0.0269
$b_{2,1}$	1.100	0.923	1.129	1.335	0.2060
$b_{2,2}$	0.300	0.148	0.326	0.505	0.1785

Table II: Parameter central estimates ($\theta_j^{cpd,\delta}$), parameter bounds ($\underline{\theta}_j^{pd,\delta}$, $\overline{\theta}_j^{pd,\delta}$) and parameter uncertainties $\Delta\theta_j^{pd,\delta}$ obtained through the partial dynamic LPV relaxation for a relaxation order $\delta = 2$

Parameter	True Value	$\underline{\theta}_j^{pd,\delta}$	$\theta_j^{cpd,\delta}$	$\overline{\theta}_j^{pd,\delta}$	$\Delta\theta_j^{pd,\delta}$
$a_{1,1}$	0.700	0.601	0.718	0.835	0.1169
$a_{2,1}$	-0.400	-0.444	-0.409	-0.373	0.0358
$a_{2,2}$	0.300	0.193	0.282	0.372	0.0894
$b_{1,1}$	0.100	0.087	0.101	0.114	0.0132
$b_{2,1}$	1.100	0.997	1.113	1.229	0.1161
$b_{2,2}$	0.300	0.169	0.297	0.424	0.1280

db and 26 db, respectively. The number of measurements N used to compute the parameter bounds is equal to 200. First, bounds on the parameters are evaluated through the static LPV approach. The obtained relaxed bounds $\underline{\theta}_j^s$, $\overline{\theta}_j^s$, the central estimate θ_j^{cs} and the parameter uncertainty bounds $\Delta\theta_j^s$, defined as $\theta_j^{cs} = \frac{\overline{\theta}_j^s + \underline{\theta}_j^s}{2}$ and $\Delta\theta_j^s = \frac{\overline{\theta}_j^s - \underline{\theta}_j^s}{2}$, are reported in Table I. Then, parameters bounds are evaluated through the partial-dynamic LPV relaxation with a relaxation order $\delta = 2$. The Matlab package SparsePOP [21] has been exploited to relax the semialgebraic problems (31) into the SDP problems (33). In Table II the obtained parameters bounds $\underline{\theta}_j^{pd,\delta}$ and $\overline{\theta}_j^{pd,\delta}$ are reported, together with the central estimate $\theta_j^{cpd,\delta}$ and the parameter uncertainty bounds $\Delta\theta_j^{pd,\delta}$ defined as $\theta_j^{cpd,\delta} = \frac{\overline{\theta}_j^{pd,\delta} + \underline{\theta}_j^{pd,\delta}}{2}$ and $\Delta\theta_j^{pd,\delta} = \frac{\overline{\theta}_j^{pd,\delta} - \underline{\theta}_j^{pd,\delta}}{2}$. Results in Tables I and II show that the true parameters are included in the parameter uncertainty intervals, as expected. Besides, the partial-dynamic LPV relaxation provides parameter bounds tighter than the ones obtained through the method proposed in [15]. As a matter of fact, even if a low value of the relaxation order δ is used, for each parameter θ_j, the uncertainty bound $\Delta\theta_j^{pd,\delta}$ is at least 25% smaller than $\Delta\theta_j^s$.

VII. Conclusion

A new technique for the evaluation of parameter uncertainty intervals for LPV systems when both the output and the scheduling signal measurements are affected by bounded noise is presented. Parameter bounds evaluation is formulated in terms of nonconvex optimization problems. In order to reduce the computational complexity of the formulated problems, the feasible set is approximated by a union of semialgebraic regions described by polynomial inequalities that involve only a small number of decision variables. Thanks to the structured sparsity of the identification problem, relaxation techniques based on linear matrix inequalities are exploited to compute parameters uncertainty intervals, which are guaranteed to contain the true parameters. The capability of the proposed identification technique to provide a less conservative estimate of parameters bounds with respect to the previously published results is shown both theoretically and by means of a numerical example.

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