Robust Pole Placement for Plants with Semialgebraic Parametric Uncertainty

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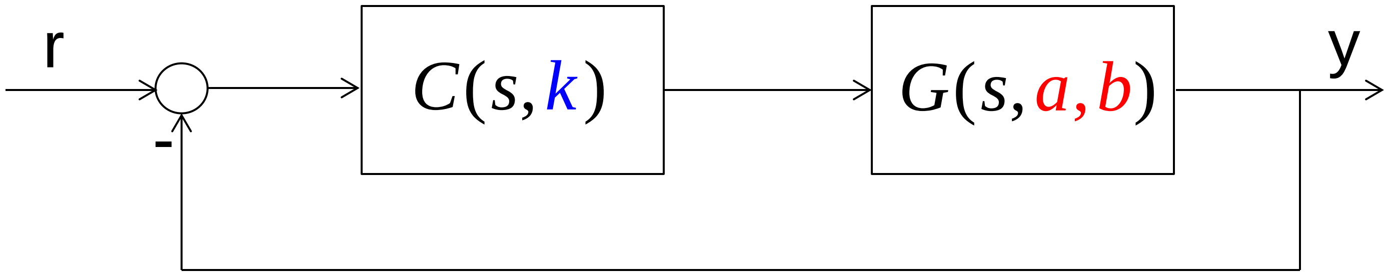
Problem Formulation

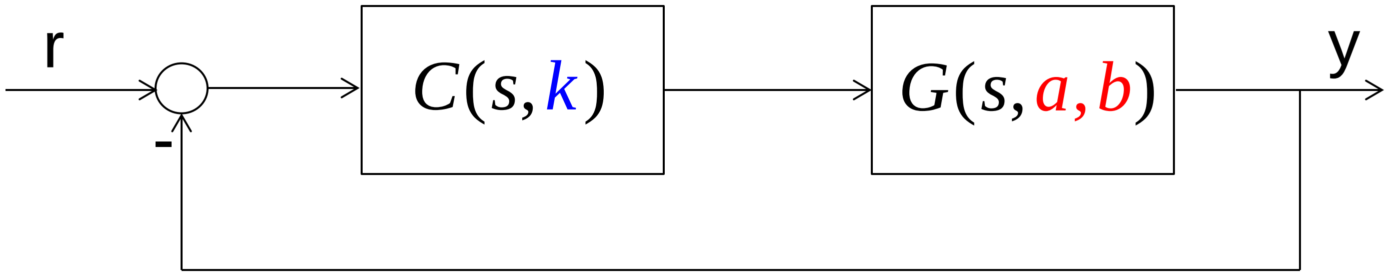


$$G(s,a,b) = \frac{N_G(s,b)}{D_G(s,a)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

(a,b) belong to an uncertainty set P

P : semialgebraic set

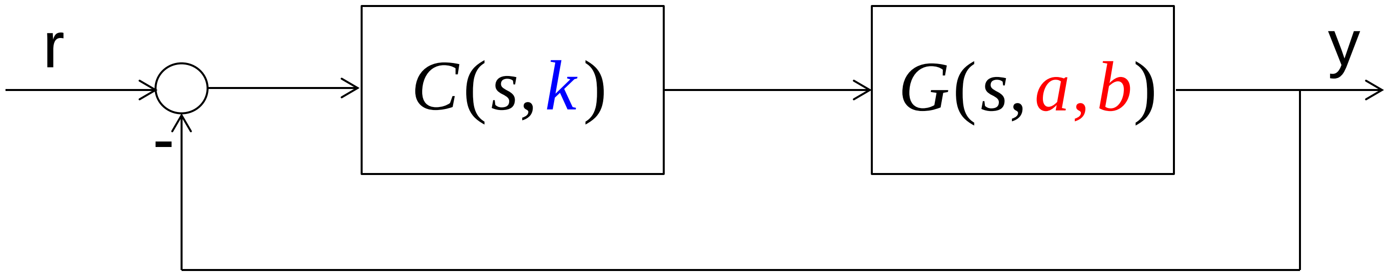
Problem Formulation



$$C(s,k)=\frac{N_C(s,k)}{D_C(s,k)}=\frac{k_r s^r+\cdots+k_1 s+k_0}{k_{2r+1}s^r+\cdots+k_{r+2}s+k_{r+1}} \qquad r\geq n-1$$

k : design parameters

Problem Formulation

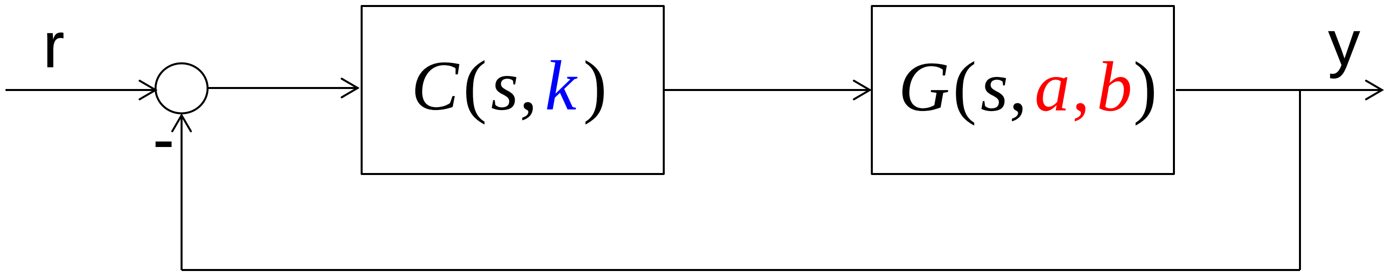


$$F(s,k,a,b) = D_G(s,a,b)D_C(s,k) + C_G(s,a,b)C_C(s,k) = f_{n+r}s^{n+r} + \cdots f_1 s + f_0$$

$$\underline{f}_j \leq f_j \leq \overline{f}_j, \qquad j = 0,\ldots n+r \qquad \text{for all } (a,b) \in P$$

$F(s,k,a,b)$: (uncertain) closed-loop characteristic polynomial

Problem Formulation



$$F(s,k,a,b) = D_G(s,a,b)D_C(s,k) + C_G(s,a,b)C_C(s,k) = f_{n+r}s^{n+r} + \cdots f_1 s + f_0$$

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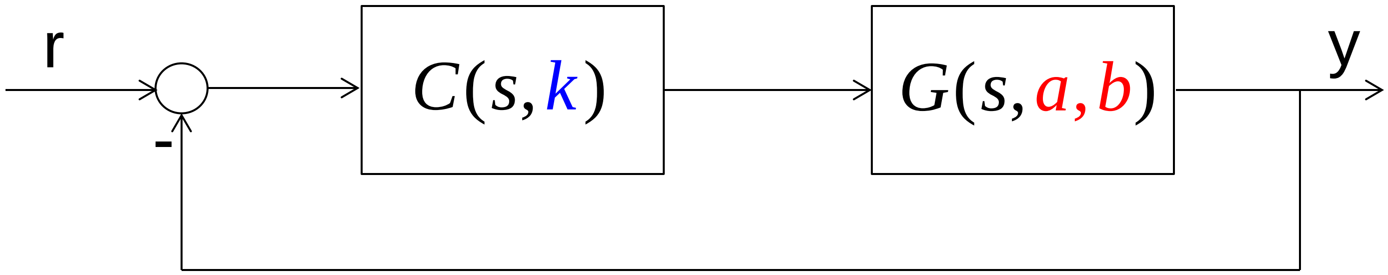
$$S(a,b)k = f, \qquad \underline{f} \le f \le \overline{f} \qquad \text{for all } (a,b) \in P$$

Problem Formulation



$$S(a,b)k = f, \qquad \underline{f} \le f \le \overline{f} \qquad \text{for all } (a,b) \in P$$

Robust Diophantine equation

Robust Pole Placement

Two problems have to be considered in robust pole placement:

- checking robust nonsingularity of the Sylvester matrix $S(a,b)$ over the uncertainty set P
- controller parameters computation

Checking robust nonsingularity of uncertain matrices

- Checking robust nonsingularity of uncertain matrices is an NP-hard problem
- Previously proposed approaches:
  - are only able to deal with the case that P is a box
  - provide conservative nonsingularity conditions
- New approach:
  - is able to deal with the case that P is a semialgebraic set
  - in the case P is a box, it provides less conservative nonsingularity conditions w.r.t. previously proposed approaches

Checking robust nonsingularity of uncertain matrices

Result 1

The uncertain matrix $S(a,b)$, with $(a,b)\in P$, is robustly nonsingular if and only if the following optimization problem is bounded:

$$h^* = \max_{\substack{x\in R^{n+r}\\ a,b\in P}} \|x\|_2^2 \qquad s.t. \qquad S(a,b)x = 0$$

Checking robust nonsingularity of uncertain matrices

$$h^* = \max_{\substack{x \in R^{n+r} \\ a,b \in P}} \|x\|_2^2 \qquad s.t. \qquad S(a,b)x = 0$$

An upper bound $\bar{h}$ of h^* can be computed by exploiting LMI-relaxation techniques for polynomial optimization problems:

- Theory of moments (*Lasserre 2001*)
- SOS decomposition (*Chesi et al. 1999; Parrillo 2003*)

Result 2

The uncertain matrix $S(a,b)$, with $(a,b) \in P$, is robustly nonsingular if $\bar{h}$ is finite

Robust nonsingularity of uncertain matrices: example

$$S(a,b)=\begin{bmatrix} 0 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_2 & b_1 & b_0 \\ 1 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 1 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 1 & a_2 & a_1 & a_0 \end{bmatrix}$$

$$b_2 \in [3-\Delta,\ 3+\Delta] \qquad a_2 \in [4-\Delta,\ 4+\Delta]$$
$$b_1 \in [8-\Delta,\ 8+\Delta] \qquad a_1 \in [3-\Delta,\ 3+\Delta]$$
$$b_0 \in [\text{-}5-\Delta,\ \text{-}5+\Delta] \qquad a_0 \in [\text{-}6-\Delta,\ \text{-}6+\Delta]$$

Obtained results: matrix $S(a,b)$ is guaranteed to be robustly nonsingular for:

- $\Delta \le 0.201$ (Beek, 1975)
- $\Delta \le 0.208$ (Jansoon and Rohn, 1999)
- $\Delta \le 0.420$ (LMI-relaxation approach)

For $\Delta = 0.427$ $S(a,b)$ is not robustly nonsingular

Design of robust controller

$$S(a,b)k = f, \qquad \underline{f} \le f \le \overline{f} \qquad \text{for all } (a,b) \in P$$

$$\underline{f} \le S(a,b)k \le \overline{f}, \qquad \text{for all } (a,b) \in P$$

$$\underline{f_i} \le S_i(a,b)k \le \overline{f_i}, \qquad i = 0,\ldots,n+r, \qquad \text{for all } (a,b) \in P$$

$$\begin{cases} S_i(a,b)k - \underline{f_i} \ge 0 \\ \overline{f_i} - S_i(a,b)k \ge 0 \end{cases} \qquad i = 0,\ldots,n+r, \qquad \text{for all } (a,b) \in P$$

Design of robust controller

Problem 1

Find the controller parameters k such that

$$\begin{cases} S_i(a,b)k - \underline{f}_i \geq 0 \\ \overline{f}_i - S_i(a,b)k \geq 0 \end{cases} \quad i = 0,\ldots,n+r, \qquad \text{for all } (a,b) \in P$$

Robust Feasibility Problem

Design of robust controller

Problem 2

Find the controller parameters k and a set of SOS polynomials $\sigma_j^2(a,b,\Theta_j)$ and $\chi_j^2(a,b,\Phi_j)$ such that

$$\begin{cases} S_i(a,b)k - \underline{f_i} = \sigma_0^2(a,b,\Theta_0) + \sum_{j=1}^{np} \sigma_j^2(a,b,\Theta_j) p_j(a,b) \\ \overline{f_i} - S_i(a,b)k = \chi_0^2(a,b,\Phi_0) + \sum_{j=1}^{np} \chi_j^2(a,b,\Phi_j) p_j(a,b) \end{cases} \quad i = 0,\ldots,n+r$$

$$P = \{(a,b) : p_j(a,b) \geq 0, \quad j = 1,\ldots np\}$$

If k is feasible for Problem 2, then k is feasible for Problem 1

Design of robust controller

Find the controller parameters k and a set of SOS polynomials $\sigma_j^2(a,b,\Theta_j)$ and $\chi_j^2(a,b,\Phi_j)$ such that

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- Enforcing $\sigma_j^2(a,b,\Theta_j)$ and $\chi_j^2(a,b,\Phi_j)$ to be SOS polynomials leads to LMI constraints in their coefficients
- Enforcing $S_i(a,b)k - \underline{f_i} = \sigma_0^2(a,b,\Theta_0) + \sum_{j=1}^{np} \sigma_j^2(a,b,\Theta_j) p_j(a,b)$ and $\overline{f_i} - S_i(a,b)k = \chi_0^2(a,b,\Phi_0) + \sum_{j=1}^{np} \chi_j^2(a,b,\Phi_j) p_j(a,b)$ leads to linear equality constraints in the design parameters k and in the coefficients of $\sigma_j^2(a,b,\Theta_j)$ and $\chi_j^2(a,b,\Phi_j)$

~~Robust~~ Feasibility Problem

Design of robust controller

Find the controller parameters k and a set of SOS polynomials $\sigma_j^2(a,b,\Theta_j)$ and $\chi_j^2(a,b,\Phi_j)$ such that

$$\begin{cases} S_i(a,b)k - \underline{f_i} = \sigma_0^2(a,b,\Theta_0) + \sum_{j=1}^{np} \sigma_j^2(a,b,\Theta_j)\, p_j(a,b) \\ \overline{f_i} - S_i(a,b)k = \chi_0^2(a,b,\Phi_0) + \sum_{j=1}^{np} \chi_j^2(a,b,\Phi_j)\, p_j(a,b) \end{cases} \qquad i = 0,\ldots,n+r$$

- Enforcing $\sigma_j^2(a,b,\Theta_j)$ and $\chi_j^2(a,b,\Phi_j)$ to be SOS polynomials leads to LMI constraints in their coefficients
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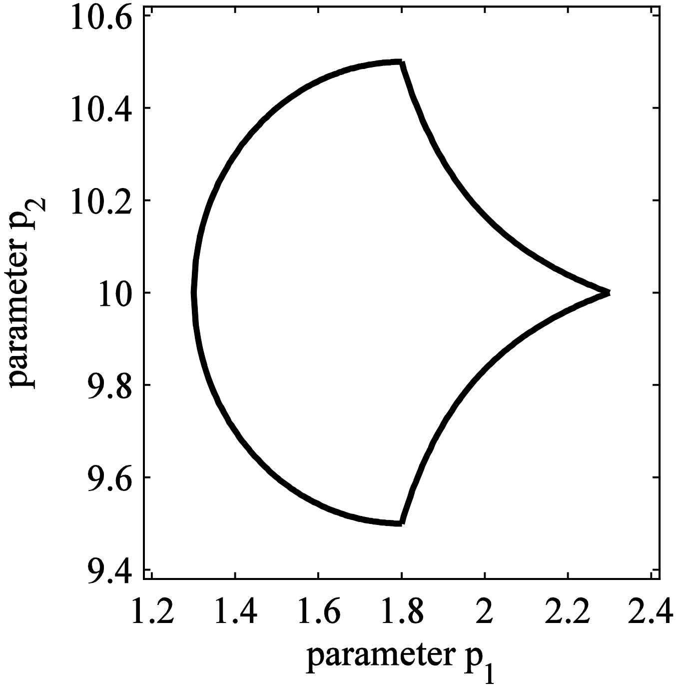
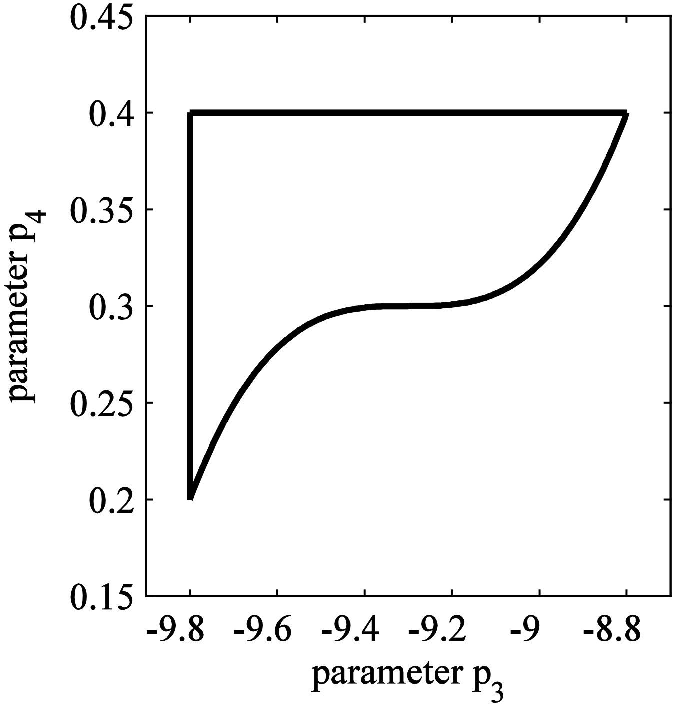
Convex Feasibility Problem

Design of robust controller
Example

$$G(s) = \frac{p_1 s + p_2}{s^2 + p_3 p_4 s + p_2}$$





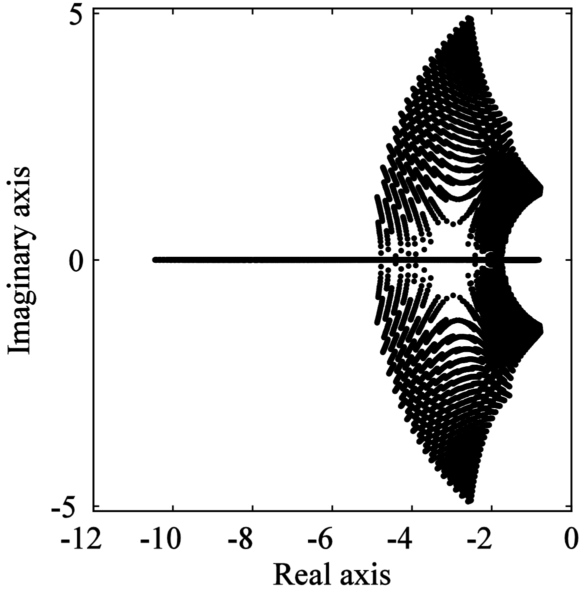
Design of robust controller
Example

$$F(s) = s^3 + f_2 s^2 + f_1 s + f_0$$

$f_2 \in [6,\ 12]$

$f_1 \in [19,\ 35]$

$f_0 \in [25,\ 29]$



$$C(s) = \frac{3.68s + 2.54}{s + 5.186}$$

Conclusions

- A novel approach to check robust nonsingularity of uncertain matrices is proposed
  - the considered problem can formulated as a polynomial optimization problem
  - the case when the entries belong to an uncertain semialgebraic set can be treated
  - in the case of interval matrices, the presented method provides less conservative nonsingularity conditions w.r.t. previously published techniques
- The problem of designing robust feedback controllers is considered
  - the design problem is reduced to a robust feasibility problem over a nonconvex uncertainty set
  - SOS representation of positive polynomials is used to compute a robust controller through convex optimization