Polytopic outer approximations of semialgebraic sets

Vito Cerone[1] and Dario Piga[2] and Diego Regruto[1]

***Abstract*— This paper deals with the problem of finding a polytopic outer approximation $\mathcal{P}^*$ of a compact semialgebraic set $\mathcal{S} \subseteq \mathbb{R}^n$. The computed polytope turns out to be an approximation of the linear hull of the set $\mathcal{S}$. The evaluation of $\mathcal{P}^*$ is reduced to the solution of a sequence of robust optimization problems with nonconvex functional, which are efficiently solved by means of convex relaxation techniques. Properties of the presented algorithm and its possible applications in the analysis, identification and control of uncertain systems are discussed.**

I. INTRODUCTION

The aim of this work is to find a polytopic outer approximation $\mathcal{P}^*$ of a compact, possibly nonconvex, semialgebraic set $\mathcal{S}$. More precisely, $\mathcal{P}^*$ should preferably be the linear hull of the set $\mathcal{S}$, or equivalently the minimum-volume polytopic outer approximation of $\mathcal{S}$. Convex polytopes are widely used in applied sciences and computational techniques, and they often play a key role in the solution of problems in mathematical programming, computational geometry, statistics or control engineering.

The main motivations of the present work come from a number of significant problems encountered in analysis, identification and control of uncertain systems. In fact, popular uncertainty models assume that the parameters describing a dynamical system belong to a given polytopic uncertainty region. For this class of uncertain systems, several methods have been developed for robust stability analysis and for designing robust controllers (see, e.g. [1], [2], [3], [4], [5], [6]). Therefore, if system parameters are known to belong to a semialgebraic set $\mathcal{S}$, a polytopic outer approximation of $\mathcal{S}$ can be constructed and the methods mentioned above can be used to efficiently deal with robust control problems.

One more significant motivating example comes from bounded-error identification of dynamical systems, where the noise corrupting the data is assumed to be bounded. In fact, one of the most challenging problem in the context of bounded-error identification is to provide a description of the so-called Feasible Parameter Set (FPS), which is the set of all system parameters consistent with the assumed model structure, measured data and error bounds (the reader can find details on the bounded-error identification approach in the survey papers [7], [8] and in the book [9]). In many cases, the FPS turns out to be a nonconvex semialgebraic set on the space of system parameters. In recent years, several algorithms have been proposed to compute an outer bounding box of the FPS for linear error-in-variables model structures [10], [11], [12], [13] and for block-oriented nonlinear systems [14], [15]. However, in some cases, it would be worth evaluating a polytopic outer approximation of the FPS, which provides a less conservative description of the FPS with respect to the description given by an outer-bounding box. Such a polytopic description can be used, for instance, to design a robust controller for the uncertain identified system.

Although several algorithms have been proposed to compute the linear hull of a set described by a finite collection of points (see e.g. [16], [17]), to the best of our knowledge no contribution can be found in literature addressing the problem of evaluating the linear hull of a semialgebraic set.

In this paper we present a novel procedure to evaluate a polytopic outer approximation of the semialgebraic set $\mathcal{S}$. The discussed algorithm makes use of results from real algebraic geometry on the representation of positive polynomials as sum-of-square (SOS) polynomials. The paper is organized as follows. Section II is devoted to the description of the problem. A general overview of the algorithm is given in Section III, while mathematical details are reported in Section IV. Finally, an illustrative example is reported in Section V.

II. PROBLEM DESCRIPTION

Let us consider a compact, possibly nonconvex, semialgebraic set $\mathcal{S}$ defined as

$$\mathcal{S} = \{x \in \mathbb{R}^n : \; g_s(x) \geq 0, \quad s = 1, \ldots, m\}, \qquad (1)$$

where $g_s(x)$ is a real-valued polynomial in the variable $x \in \mathbb{R}^n$. The aim of this work is to compute a polytopic outer approximation $\mathcal{P}^*$ of the semialgebraic set $\mathcal{S}$, i.e. $\mathcal{P}^* \supseteq \mathcal{S}$. Ideally, $\mathcal{P}^*$ should be the linear hull of the set $\mathcal{S}$, or equivalently the minimum-volume polytopic outer approximation of $\mathcal{S}$, in the sense that $\mathcal{P}^*$ is the optimizer of the following volume minimization problem

$$\begin{aligned} &\inf_{\mathcal{P} \in \mathbf{P}} \int_{\mathcal{P}} dx \\ &\text{s.t.} \\ &\mathcal{S} \subseteq \mathcal{P}, \end{aligned} \qquad (2)$$

where $\mathbf{P}$ denotes the set of all polytopes in $\mathbb{R}^n$. Although several algorithms have been proposed to compute the linear hull of a set described by a finite collection of points (see e.g. [16], [17]), to the best of our knowledge no contribution can be found in literature addressing the problem of evaluating the linear hull of a semialgebraic set. Basically, there are two main aspects which make (2) a challenging problem, i.e.

[1] V. Cerone and D. Regruto are with Dipartimento di Automatica e Informatica, Politecnico di Torino, 10129 Torino, Italy vito.cerone@polito.it diego.regruto@polito.it

[2] D. Piga is with the Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, Delft, 2628 CD, The Netherlands D.Piga@tudelft.nl

1) The linear hull of a semialgebraic set might be a polytope with an infinite number of edges, thus defined by the intersection of an infinite number of halfspaces. For instance, in the case $\mathcal{S}$ is an ellipsoid, its linear hull is described by the supporting hyperplanes at every boundary point of $\mathcal{S}$.
2) The problem of computing the exact volume $\int_{\mathcal{P}} dx$ of a polytope $\mathcal{P}$ in $\mathbb{R}^n$ is $\#P$-hard (see, e.g. [18], [19]). Although several algorithms have been proposed in literature to compute the volume of a polytope $\mathcal{P}$ through triangulation [20], [21], [22], [23], Gram's relation [24], Laplace transform [25] or randomized methods [26], [27], [28], all the approaches mentioned above require an exact description of the polytope $\mathcal{P}$ in terms of its half-space or vertex representation. Unfortunately, in (2), $\mathcal{P}$ is unknown and it has to be determined as part of the problem.

III. Polytopic outer approximation: main algorithm

In this section, we present the algorithm to compute a polytopic outer approximation $\mathcal{P}^*$ of the semialgebraic set $\mathcal{S}$ described in (1). Technical details are given in Section IV.

A. Approximation of the functional of problem (2)

As stated in the Section II, one of the main problems in solving (2) is that there is not an analytical expression for the computation of the volume of a polytope $\mathcal{P}$ in $\mathbb{R}^n$, and methods available in literature to evaluate the volume of $\mathcal{P}$ require a representation of it, while in (2) the polytope $\mathcal{P}$ is unknown. In order to overcome such a problem, a Monte Carlo sampling method is used to approximate the volume of $\mathcal{P}$. In particular, given an outer-bounding box $\mathcal{B}$ of the semialgebraic set $\mathcal{S}$ and a sequence of N random points $\{x_i\}_{i=1}^N$ uniformly distributed in $\mathcal{B}$, the integral $\int_{\mathcal{P}} dx$ is approximated by

$$\int_{\mathcal{P}} dx \approx Vol(\mathcal{B}) \frac{1}{N} \sum_{i=1}^{N} I_{\{\mathcal{P}\}}(x_i), \tag{3}$$

where $Vol(\mathcal{B})$ is the volume of the box $\mathcal{B}$ and $I_{\{\mathcal{P}\}}(x_i)$ is the indicator function of $\mathcal{P}$ defined as

$$I_{\{\mathcal{P}\}}(x_i) = \begin{cases} 1 & \text{if } x_i \in \mathcal{P} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

On the basis of (3), the volume minimization problem (2) can be approximated as

$$\begin{aligned} &\min_{\mathcal{P} \in \mathbf{P}} \sum_{i=1}^{N} I_{\{\mathcal{P}\}}(x_i) \\ &\text{s.t.} \\ &\mathcal{S} \subseteq \mathcal{P} \end{aligned} \tag{5}$$

In the following subsection, we describe how to compute a polytope $\mathcal{P}^*$ minimizing problem (5).

B. Description of the algorithm

The key steps of the procedure proposed in this paper to compute the polytopic outer-approximation $\mathcal{P}^*$ of the semialgebraic set $\mathcal{S}$ are given by the following algorithm.

Algorithm 1: **Polytopic outer approximation $\mathcal{P}^*$ of $\mathcal{S}$**

A1.1 Generate a list $\mathcal{L} = \{x_i\}_{i=1}^N$ of N random points uniformly distributed in $\mathcal{B}$.
A1.2 Set $j = 1$.
A1.3 Compute an half-space $\mathcal{H}_j$, defined as $\mathcal{H}_j : \omega^{(j)^{\mathrm{T}}} x + b^{(j)} \geq 0$, that contains the minimum number of points of the list $\mathcal{L}$ and such that $\mathcal{S} \subseteq \mathcal{H}_j$, i.e.

$$\begin{aligned} \omega^{(j)}, b^{(j)} = &\arg \min_{\substack{\omega \in \mathbb{R}^n \\ b \in \mathbb{R}}} \sum_{i=1}^{N} I_{\{\mathcal{H}_j\}}(x_i) \\ &\text{s.t.} \\ &\omega \neq 0 \\ &\omega^{\mathrm{T}} x + b \geq 0 \quad \forall x \in \mathcal{S} \\ &x_i \in \mathcal{L}, \quad i = 1, \ldots, N \end{aligned} \tag{6}$$

A1.4 Collect all the points $x_i \in \mathcal{L}$ belonging to the half-space $\mathcal{H}_j$ in a list $\mathcal{L}_j$. Let N_j be the number of elements of $\mathcal{L}_j$.
A1.5 If $N_j < N$, then $\mathcal{L} \leftarrow \mathcal{L}_j$, $N \leftarrow N_j$, $j \leftarrow j + 1$ and go to step A1.3. On the other hand, if $N_j = N$, then set $J = j - 1$ and go to step A1.6.
A1.6 Define the polytope $\mathcal{P}^*$ as

$$\mathcal{P}^* = \mathcal{B} \cap \bigcap_{j=1}^{J} \mathcal{H}_j \tag{7}$$

■

Algorithm 1 generates a sequence of half-spaces $\mathcal{H}_1, \ldots, \mathcal{H}_J$ as follows. First, the half-space $\mathcal{H}_1$ that minimize the area of the polytope $\mathcal{B} \cap \mathcal{H}_1$ is computed. The area of $\mathcal{B} \cap \mathcal{H}_1$, given by the integral $\int_{\mathcal{B} \cap \mathcal{H}_1} dx$, is approximated (up to the constant $Vol(\mathcal{B})/N$) by $\sum_{i=1}^{N} I_{\{\mathcal{H}_1\}}(x_i)$ (corresponding to the objective function of problem (6)), where $\{x_i\}_{i=1}^N$ is a sequence of random points uniformly distributed in $\mathcal{B}$. Then, the new half-space $\mathcal{H}_2$ that minimizes the area of the polytope $\mathcal{B} \cap \mathcal{H}_1 \cap \mathcal{H}_2$ is generated. In order to approximate the area of $\mathcal{B} \cap \mathcal{H}_1 \cap \mathcal{H}_2$, all the points x_i of the list $\mathcal{L} = \{x_i\}_{i=1}^N$ that do not belong to the polytope $\mathcal{B} \cap \mathcal{H}_1$ are discarded, and all and only the points belonging to $\mathcal{B} \cap \mathcal{H}_1$ are collected in a new list $\mathcal{L}_1 = \{x_i\}_{i=1}^{N_1}$ (step A1.4). The area of $\mathcal{B} \cap \mathcal{H}_1 \cap \mathcal{H}_2$ is then approximated by $\sum_{i=1}^{N_1} I_{\{\mathcal{H}_2\}}(x_i)$, with $x_i \in \mathcal{L}_1$. The procedure is repeated until $N_{J+1} = N_J$ (step A1.5), which means that the number of samples x_i belonging to the polytope $\mathcal{B} \cap \mathcal{H}_1 \cap \ldots \cap \mathcal{H}_{J+1}$ is equal to the number of samples x_i belonging to the polytope $\mathcal{B} \cap \mathcal{H}_1 \cap \ldots \cap \mathcal{H}_J$. Note that, because of the constraint

$\omega^{\mathrm{T}} x + b \geq 0 \quad \forall x \in \mathcal{S}$ appearing in optimization problem (6), half-spaces $\mathcal{H}_1, \ldots, \mathcal{H}_J$ are guaranteed to contain the semialgebraic set $\mathcal{S}$, and thus $\mathcal{P}^* = \mathcal{B} \cap \bigcap_{j=1}^{J} \mathcal{H}_j$ is an outer approximation of $\mathcal{S}$. Technical details of step A1.3, which is the core of Algorithm 1, are provided in the following section.

***Remark** 1:* An outer-bounding box $\mathcal{B}$ of the semialgebraic set can be evaluated by computing the minimum and maximum value of each component of the vector x over the semialgebraic set $\mathcal{S}$, that is by solving the polynomial optimization problems

$$\underline{x}^{(k)} = \min_{x \in \mathbb{R}^n} x^{(k)} \quad s.t. \quad x \in \mathcal{S}, \quad k = 1, \ldots, n; \tag{8a}$$

$$\overline{x}^{(k)} = \max_{x \in \mathbb{R}^n} x^{(k)} \quad s.t. \quad x \in \mathcal{S}, \quad k = 1, \ldots, n, \tag{8b}$$

where $x^{(k)}$ denotes the k-th component of vector x. A lower and an upper bound of $\underline{x}^{(k)}$ and $\overline{x}^{(k)}$, respectively, can be then computed by exploiting the techniques presented in [29], [30], [31] to relax a polynomial optimization problem into a sequence of semidefinite programming (SDP) problems. ■

IV. A CONVEX RELAXATION APPROACH TO EVALUATE THE OPTIMAL HALF-SPACE $\mathcal{H}_j$

In this section we show how the robust optimization problem (6) can be efficiently solved by means of convex relaxations.

A. Approximation of the objective function

Note that the functional of problem (6) is nonconvex since it is the sum of the indicator functions $I_{\{\mathcal{H}_j\}}(x_i)$ defined as

$$I_{\{\mathcal{H}_j\}}(x_i) = \begin{cases} 1 & \text{if } \omega^{\mathrm{T}} x_i + b \geq 0 \\ 0 & \text{if } \omega^{\mathrm{T}} x_i + b < 0 \end{cases} \tag{9}$$

Each indicator function $I_{\{\mathcal{H}_j\}}(x_i)$ is then approximated by the convex function $R_{\{\mathcal{H}_j\}}(x_i)$ defined as

$$R_{\{\mathcal{H}_j\}}(x_i) = \begin{cases} \omega^{\mathrm{T}} x_i + b & \text{if } \omega^{\mathrm{T}} x_i + b \geq 0 \\ 0 & \text{if } \omega^{\mathrm{T}} x_i + b < 0 \end{cases} \tag{10}$$

Functions $I_{\{\mathcal{H}_j\}}(x_i)$ and $R_{\{\mathcal{H}_j\}}(x_i)$ are plotted in Fig. 1. Problem (6) is then relaxed by the following optimization problem with convex functional:

$$\begin{aligned} \omega^{(j,*)}, b^{(j,*)} = & \arg \min_{\substack{\omega \in \mathbb{R}^n \\ b \in \mathbb{R}}} \sum_{i=1}^{N} R_{\{\mathcal{H}_j\}}(x_i) \\ & \text{s.t.} \\ & \omega \neq 0 \\ & \omega^{\mathrm{T}} x + b \geq 0 \quad \forall x \in \mathcal{S} \\ & x_i \in \mathcal{L}, \quad i = 1, \ldots, N \end{aligned} \tag{11}$$

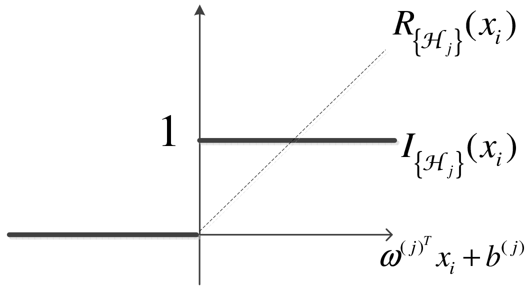

Fig. 1. Indicator function $I_{\{\mathcal{H}_j\}}(x_i)$ (solid line) and approximate function $R_{\{\mathcal{H}_j\}}(x_i)$ (dashed line). When $\omega^{(j)^{\mathrm{T}}} x_i + b^{(j)} < 0$, $I_{\{\mathcal{H}_j\}}(x_i)$ and $R_{\{\mathcal{H}_j\}}(x_i)$ are overlapped and they are equal to 0.

***Remark** 2:* It is worth remarking that the idea of approximating the nonconvex functional $\sum_{i=1}^{N} I_{\{\mathcal{H}_j\}}(x_i)$ with the convex functional $\sum_{i=1}^{N} R_{\{\mathcal{H}_j\}}(x_i)$ turns out to be similar to the relaxation of the ℓ_0-quasi-norm with the ℓ_1-norm in the computation of the sparsest solution of a system of linear equations with more unknowns than constraints (see, e.g., [32], [33]). More precisely, given a set of equations $Ay = B$, with $A \in \mathbb{R}^{p,n}$, $B \in \mathbb{R}^p$ and $p < n$, one has to find $y \in \mathbb{R}^n$ solving the following minimization problem:

$$\min_{y \in \mathbb{R}^n} \|y\|_{\ell_0} \quad \text{s.t.} \quad Ay = B. \tag{12}$$

The nonconvex optimization problem (12) is then relaxed by the linear programming (LP) problem

$$\min_{y \in \mathbb{R}^n} \|y\|_{\ell_1} \quad \text{s.t.} \quad Ay = B. \tag{13}$$

Equivalently, by introducing the slack variable $t = [t_1, \ldots, t_n]^{\mathrm{T}}$, problem (12) can be rewritten as

$$\begin{aligned} & \min_{t \in \mathbb{R}^n} \sum_{i=1}^{n} t_i \\ & \text{s.t.} \\ & Ay = B \\ & t_i = \begin{cases} 1 & \text{if } y_i \neq 0 \\ 0 & \text{if } y_i = 0 \end{cases} \qquad i = 1, \ldots, n \end{aligned} \tag{14}$$

and then relaxed as

$$\begin{aligned} & \min_{r \in \mathbb{R}^n} \sum_{i=1}^{n} r_i \\ & \text{s.t.} \\ & Ay = B \\ & r_i = |y_i| \qquad i = 1, \ldots, n. \end{aligned} \tag{15}$$

This means that every term t_i of the objective function in problem (14) is approximated in (15) by the function $r_i = |y_i|$, whose plot is reported in Fig. 2, together with the plot of the function t_i. The interested reader is referred, for instance, to works [34], [35], [36], where the

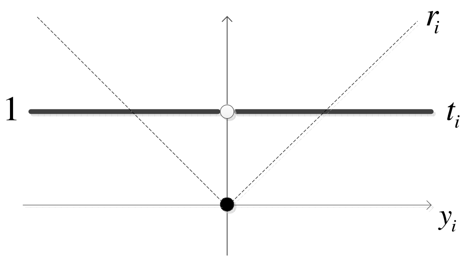

Fig. 2. Nonconvex function t_i (solid line) and convex function $r_i = |y_i|$. The function t_i is equal to 1 for all $y_i \in \mathbb{R} : y_i \neq 0$, while when $y_i = 0$, then $t_i = 0$.

approach of relaxing the ℓ_0-quasi-norm into the ℓ_1-norm is successfully applied in the field of compressive sensing and in identification for model structure selection. ■

B. Approximation of semi-infinite constraints

Let us go back to optimization problem (11). Now, the problem of treating the semi-infinite constraint $\omega^{\mathrm{T}} x + b \geq 0 \quad \forall x \in \mathcal{S}$ arises. By exploiting results from real algebraic geometry on the representation of positive polynomials over semialgebraic sets as sum-of-square (SOS) polynomials, problem (11) is replaced by the following problem:

$$\begin{aligned} \omega^{(j,\delta)}, b^{(j,\delta)} = \arg \min_{\substack{\omega \in \mathbb{R}^n \\ b \in \mathbb{R}}} \quad & \sum_{i=1}^{N} R_{\{\mathcal{H}_j\}}(x_i) \\ \text{s.t.} \quad & \\ & \omega \neq 0, \\ & \omega^{\mathrm{T}} x + b = \sigma_0 + \sum_{s=1}^{m} \sigma_s g_s(x), \\ & \sigma_0, \sigma_1, \ldots, \sigma_m \in \Sigma[x], \\ & deg(\sigma_0), deg(\sigma_1 g_1(x)), \ldots, deg(\sigma_m g_m(x)) \leq 2\delta, \end{aligned} \tag{16}$$

where $\Sigma[x]$ denotes the space of sum-of-square (SOS) polynomials in the variable x and $\delta \in \mathbb{N}$ is a given integer such that $2\delta \geq \max\{deg(g_1(x)), \ldots, deg(g_m(x))\}$. In order to cope with the constraint $\omega \neq 0$ appearing in (16), problem (16) is splitted into the two following normalized problems:

$$\begin{aligned} \overline{\omega}^{(j,\delta)}, \overline{b}^{(j,\delta)} = \arg \min_{\substack{\omega \in \mathbb{R}^n \\ b \in \mathbb{R}}} \quad & \sum_{i=1}^{N} R_{\{\mathcal{H}_j\}}(x_i) \\ \text{s.t.} \quad & \\ & \omega_1 = 1, \\ & \omega^{\mathrm{T}} x + b = \sigma_0 + \sum_{s=1}^{m} \sigma_s g_s(x), \\ & \sigma_0, \sigma_1, \ldots, \sigma_m \in \Sigma[x], \\ & deg(\sigma_0), deg(\sigma_1 g_1(x)), \ldots, deg(\sigma_m g_m(x)) \leq 2\delta, \end{aligned} \tag{17a}$$

$$\begin{aligned} \underline{\omega}^{(j,\delta)}, \underline{b}^{(j,\delta)} = \arg \min_{\substack{\omega \in \mathbb{R}^n \\ b \in \mathbb{R}}} \quad & \sum_{i=1}^{N} R_{\{\mathcal{H}_j\}}(x_i) \\ \text{s.t.} \quad & \\ & \omega_1 = -1, \\ & \omega^{\mathrm{T}} x + b = \sigma_0 + \sum_{s=1}^{m} \sigma_s g_s(x), \\ & \sigma_0, \sigma_1, \ldots, \sigma_m \in \Sigma[x], \\ & deg(\sigma_0), deg(\sigma_1 g_1(x)), \ldots, deg(\sigma_m g_m(x)) \leq 2\delta, \end{aligned} \tag{17b}$$

where ω_1 denotes the first component of vector ω. The optimizer $\{\omega^{(j,\delta)}, b^{(j,\delta)}\}$ of problem (16) is the given by the pair $\{\overline{\omega}^{(j,\delta)}, \overline{b}^{(j,\delta)}\}$ or $\{\underline{\omega}^{(j,\delta)}, \underline{b}^{(j,\delta)}\}$ that provides the minimum value of the objective function $\sum_{i=1}^{N} R_{\{\mathcal{H}_j\}}(x_i)$.

In order to avoid confusion, it is worth stressing that only ω, b and the coefficients of the polynomials $\sigma_0, \sigma_1, \ldots, \sigma_m$ are decision variables of problems (17), while x is not an optimization variable.

***Property** 1:* Problems (17) are convex. In fact, checking if the polynomial $\omega^{\mathrm{T}} x + b$ is equal to $\sigma_0 + \sum_{s=1}^{m} \sigma_s g_s(x)$ leads to linear equalities in $\omega^{\mathrm{T}} x + b$ and in the unknown coefficients of the polynomials $\sigma_0, \sigma_1, \ldots, \sigma_m$. Besides, enforcing $\sigma_0, \sigma_1, \ldots, \sigma_m$ to be sum of square polynomials leads to linear matrix inequality (LMI) constraints in the coefficients of $\sigma_0, \sigma_1, \ldots, \sigma_m$.

***Property** 2:* The semialgebraic set $\mathcal{S}$ is guaranteed to belong to the half-space $\mathcal{H}_j : \omega^{(j,\delta)^{\mathrm{T}}} x + b^{(j,\delta)} \geq 0$, i.e.

$$\mathcal{S} \subseteq \mathcal{H}_j. \tag{18}$$

Proof: Indeed, the optimal solution $\omega^{(j,\delta)}, b^{(j,\delta)}$ of problem (16) is such that $\omega^{(j,\delta)^{\mathrm{T}}} x + b^{(j,\delta)} = \sigma_0 + \sum_{s=1}^{m} \sigma_s g_s(x)$ for some SOS polynomials $\sigma_0, \sigma_1, \ldots, \sigma_m$. From the definition of the semialgebraic set $\mathcal{S}$ in (1), $g_1(x), \ldots, g_m(x) \geq 0$ for all x belonging to $\mathcal{S}$. Therefore, for all $x \in \mathcal{S}$, the two terms of the equations $\omega^{(j,\delta)^{\mathrm{T}}} x + b^{(j,\delta)} = \sigma_0 + \sum_{s=1}^{m} \sigma_s g_s(x)$ are always nonnegative since $\sigma_0, \sigma_1, \ldots, \sigma_m$ are sum of square polynomials, hence nonnegative. This means that, for all $x \in \mathcal{S}$, $\omega^{(j,\delta)^{\mathrm{T}}} x + b^{(j,\delta)} \geq 0$, then $\mathcal{S} \subseteq \mathcal{H}_j$. ■

***Remark** 3:* On the basis of *Putinar's Positivstellensatz* [37], a polynomial $f(x)$ which is positive over a compact semialgebraic set $\mathcal{S}$ can be written as

$$\begin{cases} f(x) = \sigma_0 + \sum_{s=1}^{m} \sigma_s g_s(x) \\ \text{for some } \sigma_0, \sigma_1, \ldots, \sigma_m \in \Sigma[x] \end{cases} \tag{19}$$

Therefore, the term $\omega^{(j,*)^{\mathrm{T}}} x + b^{(j,*)}$ given by the optimal solution of the robust optimization problem (11) can be

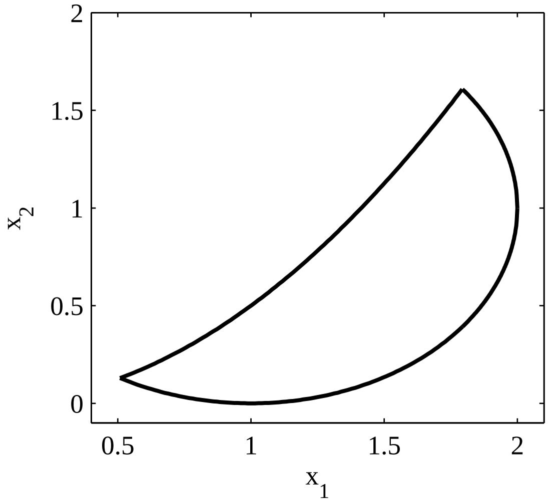

Fig. 3. Nonconvex semialgebraic set $\mathcal{S}$.

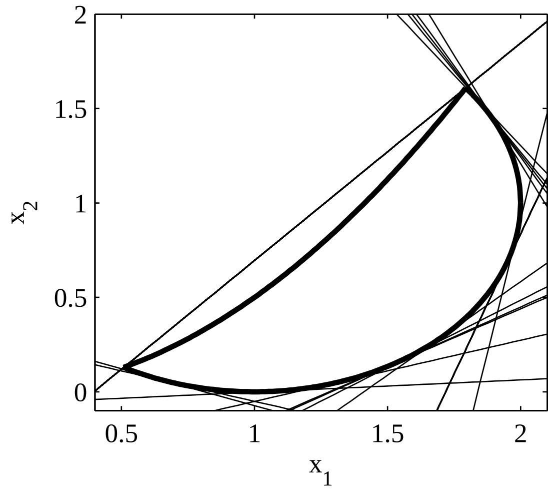

Fig. 4. Nonconvex semialgebraic set $\mathcal{S}$ (thick line) and half-spaces defining the polytopic outer approximation $\mathcal{P}^*$ (thin lines).

written as

$$\begin{cases} \omega^{(j,*)^{\mathrm{T}}} x + b^{(j,*)} = \sigma_0 + \sum_{s=1}^{m} \sigma_s g_s(x) \\ \text{for some } \sigma_0, \sigma_1, \ldots, \sigma_m \in \Sigma[x] \end{cases} \tag{20}$$

This means that, although conservativeness is introduced by replacing the robust optimization problem (11) with the convex problems (17), problem (16) provides the optimal solution $\omega^{(j,*)^{\mathrm{T}}} x,\ b^{(j,*)}$ for δ large enough. Nevertheless, in practice, the conservative solution $\omega^{(j,\delta)^{\mathrm{T}}} x,\ b^{(j,\delta)}$ and the optimal one $\omega^{(j,*)^{\mathrm{T}}} x,\ b^{(j,*)}$ coincide with each other for small values of δ. ■

V. AN ILLUSTRATIVE EXAMPLE

An illustrative example is presented in this section in order to show the capabilities of the proposed algorithm. The considered semialgebraic set $\mathcal{S}$ is the two-dimensional nonconvex region plotted in Fig. 3 described by:

$$\begin{aligned} \mathcal{S} = \{ & (x_1, x_2) \in \mathbb{R}^2 : \\ & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1; \quad x_2 \leq 0.5 x_1^2 \} . \end{aligned} \tag{21}$$

First, the outer-bounding box $\mathcal{B} = \{(x_1, x_2) \in \mathbb{R}^2 : 0.46 \leq |x_1| \leq 2.02;\ -0.03 \leq |x_2| \leq 1.64\}$ has been evaluated by means of the techniques discussed in Remark 1. Then, a sequence of $N = 100$ random points uniformly distributed in $\mathcal{B}$ has been generated and a polytopic outer approximation P^δ of the set $\mathcal{S}$ has been evaluated through Algorithm 1 and by replacing the robust optimization problem (6) with the convex problems (17). The half-spaces defining the computed polytope P^* are reported in Fig. 4, which shows that the nonconvex set $\mathcal{S}$ is contained in P^*, as expected.

VI. CONCLUSIONS

An algorithm for computing an approximation $\mathcal{P}^*$ of the linear hull of a nonconvex semialgebraic set is proposed in the paper. The half-spaces defining $\mathcal{P}^*$ are computed by solving a collection of robust optimization problems with nonconvex functional, which is efficiently approximated as a sum of convex functions. The approximation of the nonconvex functional turns out to be similar to the relaxation of the ℓ_0-quasi-norm with the ℓ_1-norm in the computation of the sparsest solution of a system of linear equations with more unknowns than constraints. Decomposition of positive polynomials over a semialgebraic set as sum-of-square polynomials is then exploited in order to reformulate semi-infinite constraints in terms of convex linear-matrix-inequality constraints. The presented algorithm can be efficiently employed, for instance, to design robust controllers for plants with polynomial parametric uncertainty or in bounded-error identification to compute an outer approximation of the set of all system parameters consistent with the measured data.

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