Polytopic outer-approximation of semialgebraic sets

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Semialgebraic sets

Definition

A semialgebraic set is a subset of $\mathbb{R}^n$ defined by a finite sequence of polynomial equality and inequality constraints.

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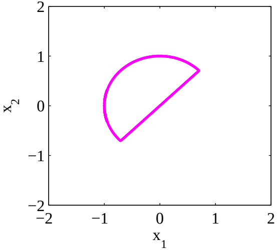

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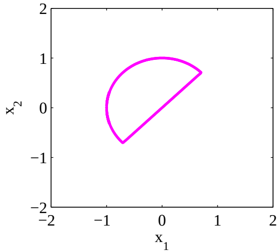

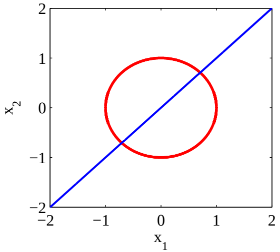

$$x_1^2 + x_2^2 \leq 1$$
$$x_2 \geq x_1$$

Polytopic outer-approximation

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An Euclidean set $\mathcal{P}$ is called a polytopic outer-approximation of $\mathcal{S}$ if $\mathcal{P}$ is a polytope and $\mathcal{S} \subseteq \mathcal{P}$.

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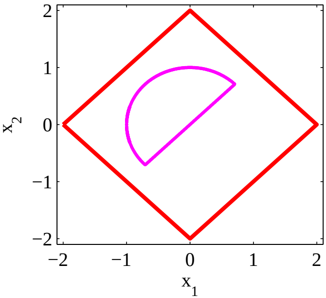

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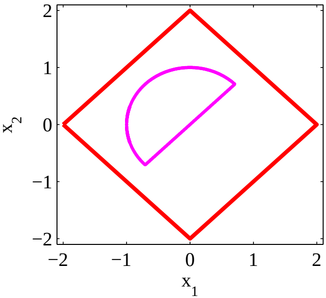

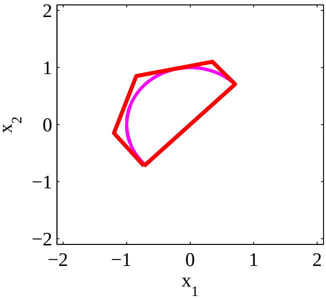

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Question

What is a "reasonable" indicator to measure the level of accuracy in approximating a set?

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Volume of the polytopic outer-approximation $\mathcal{P}$, i.e. $\int_{\mathcal{P}} 1 \mathrm{d}x$.

Problem formulation

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Among all the polytopes $\mathcal{P}$ containing $\mathcal{S}$ find the one with minimum volume,

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Problem

Find the liner hull of $\mathcal{S}$.

A motivation example

Design of Robust Controllers for Uncertain LTI Systems

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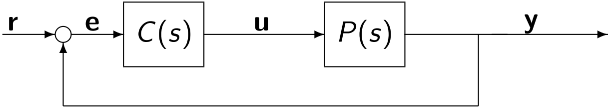

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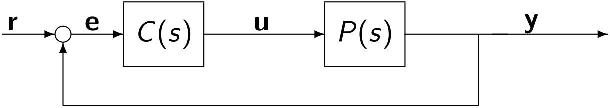

$$P(s) = \frac{s+1}{s^2 + p_1 s + p_2}$$

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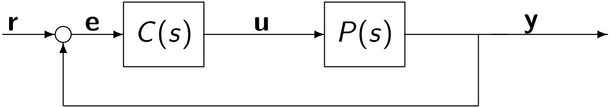

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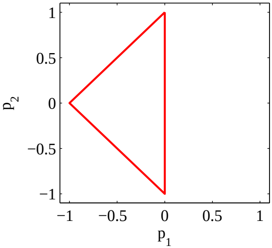

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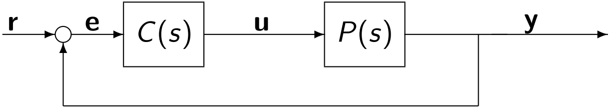

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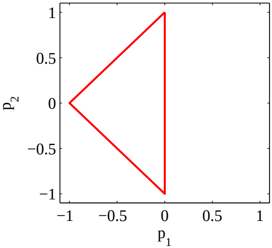

Well-settled techniques to design robust controllers if $p_1, p_2 \in \mathcal{P}$

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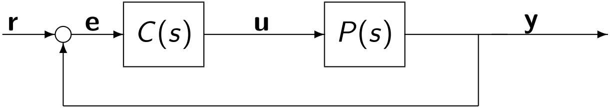

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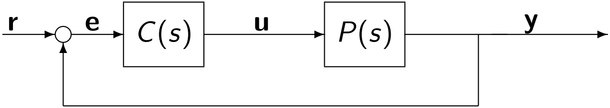

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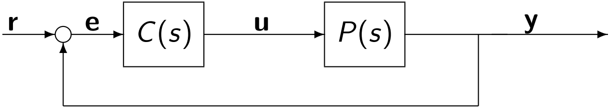

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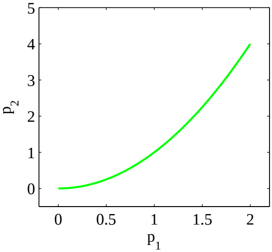

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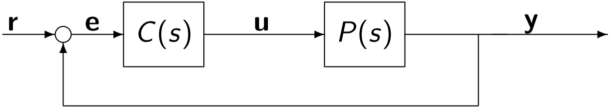

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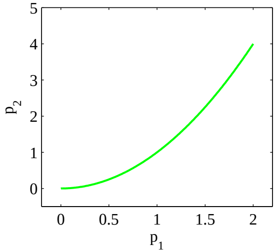

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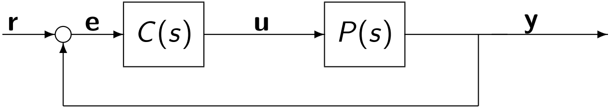

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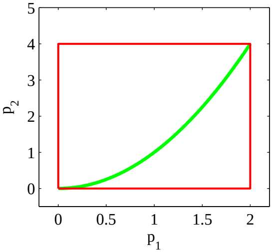

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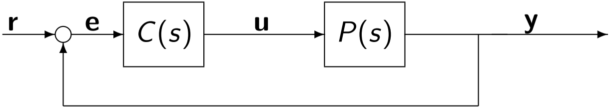

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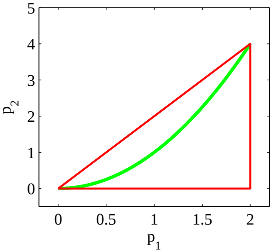

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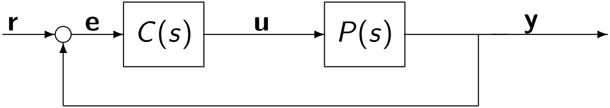

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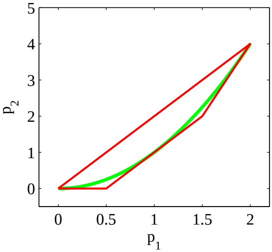

Volume Computation

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Computing the volume of a polytope is an NP-hard problem.

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Trouble 2

We don't know $\mathcal{P}$!

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1. Take an outer-bounding box $\mathcal{B}$ of the Euclidean set $\mathcal{S}$

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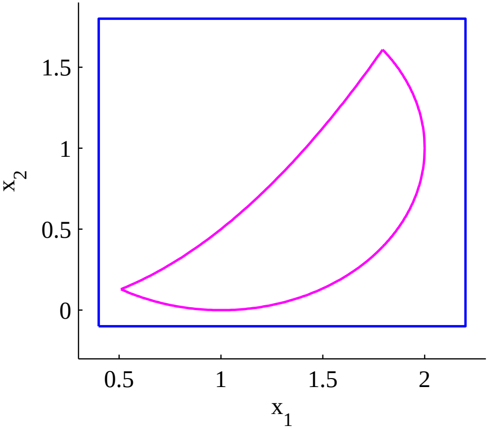

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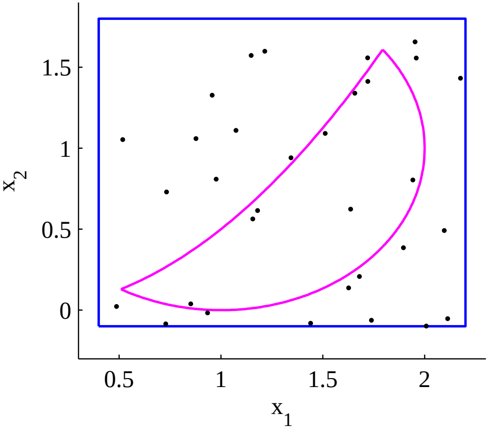

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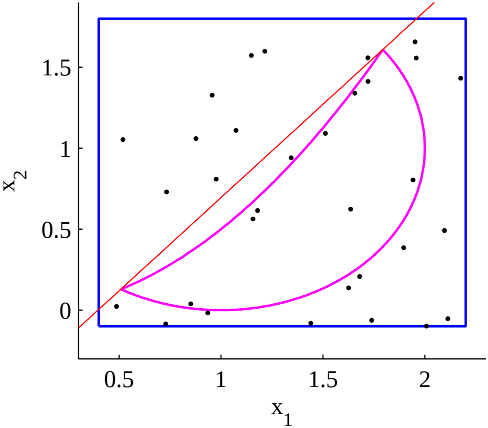

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4. Up-to-date the list $\mathcal{L}$ by getting rid of all the points that do not belong to $\mathcal{H}$ and go to step 3

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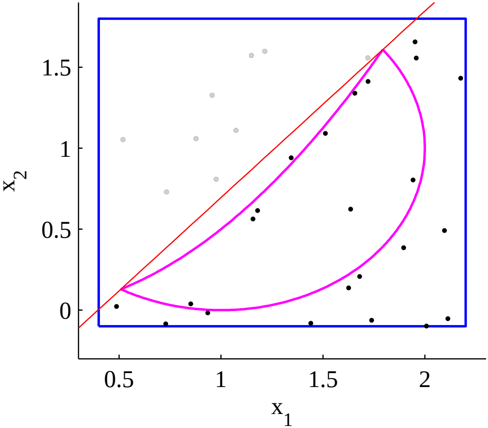

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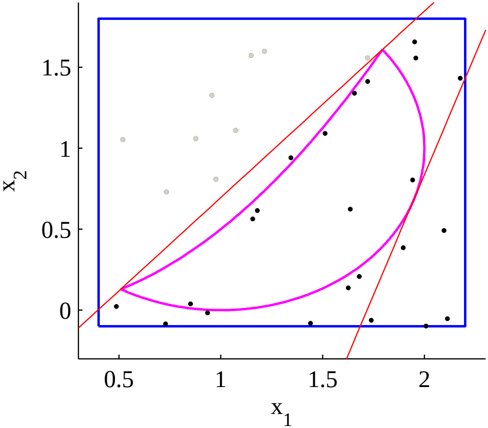

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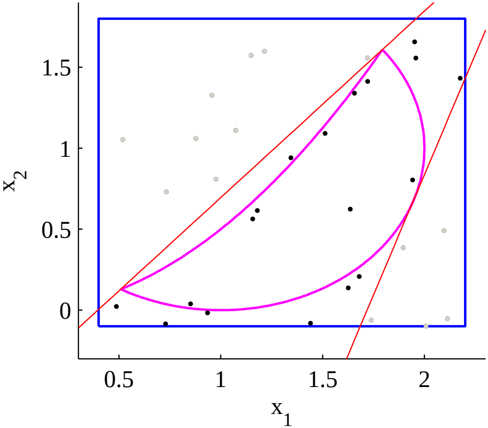

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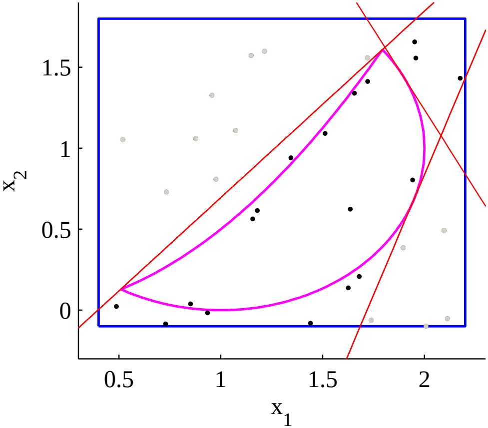

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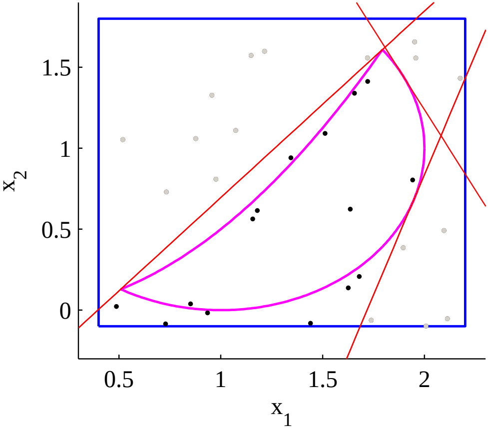

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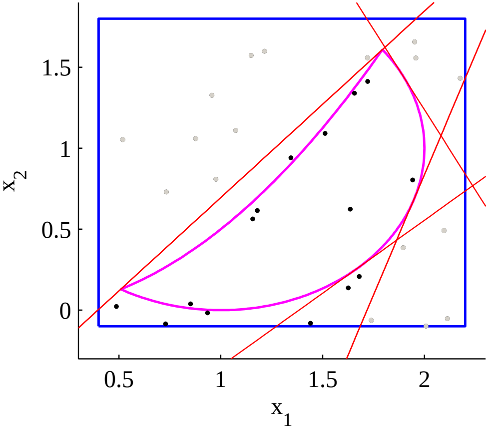

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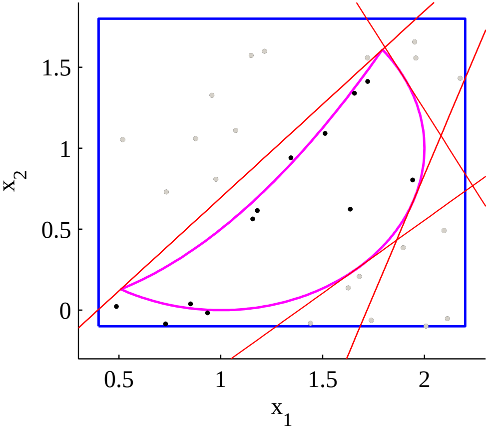

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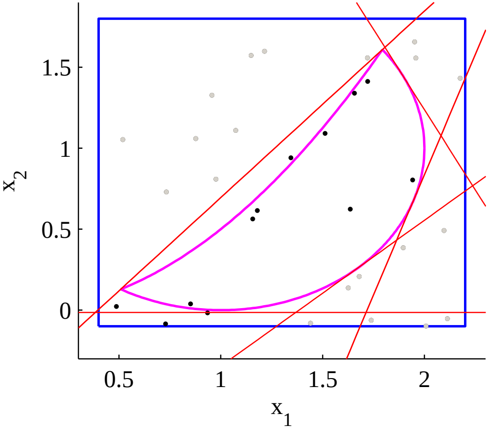

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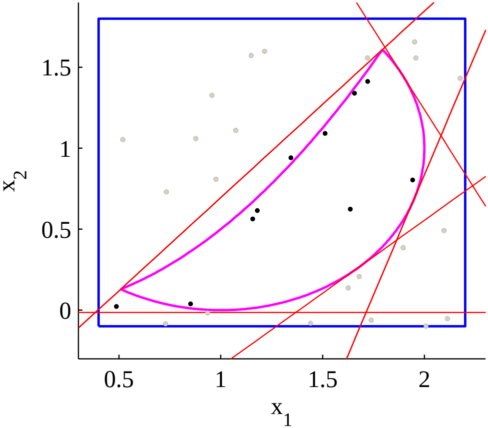

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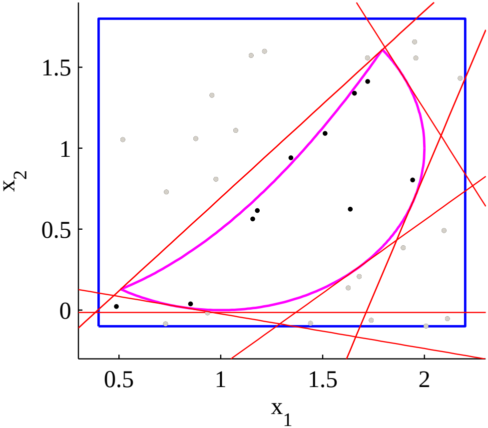

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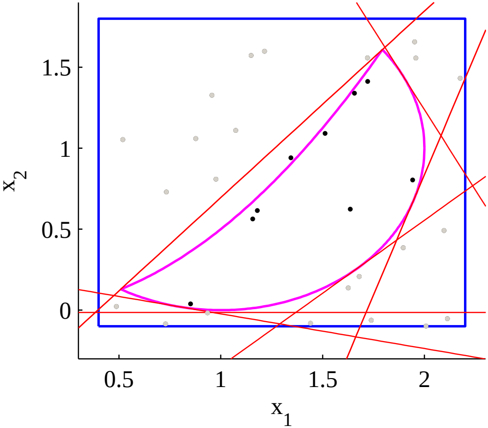

Computation of the half-spaces

Compute the half-space $\mathcal{H} : w^\top x + b \geq 0$ containing the minimum number of points of the list $\mathcal{L}$ and such that $\mathcal{S} \subseteq \mathcal{H}$, i.e.

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$$
\begin{aligned}
&\min_{w,b} \sum_{i=1}^{N} \mathcal{I}_{\mathcal{H}}(x_i) \quad x_i \in \mathcal{L} \\
&\text{s.t.} \quad w^\top x + b \geq 0 \;\; \forall x \in \mathcal{S}
\end{aligned}
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$$\mathcal{I}_{\{\mathcal{H}\}}(x_i) = \begin{cases} 1 & \text{if } w^\top x_i + b \geq 0 \\ 0 & \text{if } w^\top x_i + b < 0 \end{cases}$$

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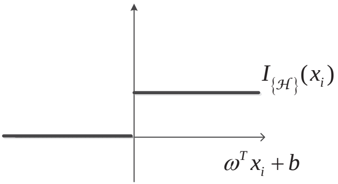

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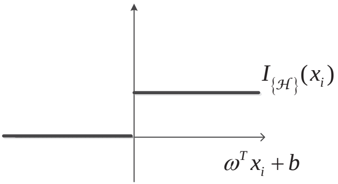

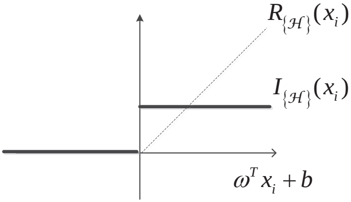

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$$\mathcal{S} = \{x : g_j(x) \geq 0, j = 1, \ldots, M\}$$

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$$\min_{\substack{w,\, b \\ \Theta_0, \ldots, \Theta_M}} \quad \sum_{i=1}^{N} \mathcal{R}_{\mathcal{H}}(x_i) \quad x_i \in \mathcal{L}$$

$$\text{s.t.} \quad w^\top x + b = \sigma_0^2(x, \Theta_0) + \sigma_1^2(x, \Theta_1) g_1(x) + \ldots + \sigma_M^2(x, \Theta_M) g_M(x)$$

Computation of the half-spaces

$$\min_{w,b} \sum_{i=1}^{N} \mathcal{R}_{\mathcal{H}}(x_i) \quad x_i \in \mathcal{L}$$

$$\text{s.t.} \quad w^\top x + b \geq 0 \;\; \forall x \in \mathcal{S}$$

$$\mathcal{S} = \{x : g_j(x) \geq 0, j = 1, \ldots, M\}$$

$$\min_{\substack{w,\, b \\ \Theta_0, \ldots, \Theta_M}} \sum_{i=1}^{N} \mathcal{R}_{\mathcal{H}}(x_i) \quad x_i \in \mathcal{L}$$

$$\text{s.t.} \quad w^\top x + b = \sigma_0^2(x,\Theta_0) + \sigma_1^2(x,\Theta_1) g_1(x) + \ldots + \sigma_M^2(x,\Theta_M) g_M(x)$$

$$\sigma_0^2(x,\Theta_0), \sigma_1^2(x,\Theta_1), \ldots, \sigma_M^2(x,\Theta_M) \quad \text{SOS}$$

Computation of the half-spaces

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$$\sigma_0^2(x, \Theta_0), \sigma_1^2(x, \Theta_1), \ldots, \sigma_M^2(x, \Theta_M) \quad \text{SOS}$$

~~Robust~~ Optimization problem

TUDelft TU/e

Computation of the half-spaces

$$\min_{w,b} \sum_{i=1}^{N} \mathcal{R}_{\mathcal{H}}(x_i) \quad x_i \in \mathcal{L}$$

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$$\mathcal{S} = \{x : g_j(x) \geq 0, j = 1, \ldots, M\}$$

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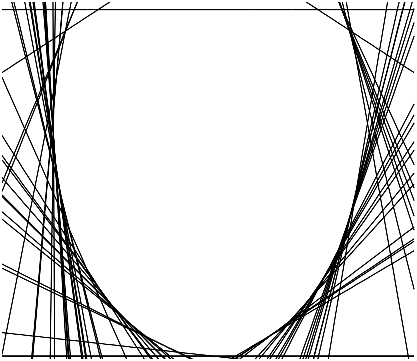
$$\text{s.t.} \quad w^\top x + b = \sigma_0^2(x,\Theta_0) + \sigma_1^2(x,\Theta_1) g_1(x) + \ldots + \sigma_M^2(x,\Theta_M) g_M(x)$$

$$\sigma_0^2(x,\Theta_0), \sigma_1^2(x,\Theta_1), \ldots, \sigma_M^2(x,\Theta_M) \quad \text{SOS}$$

Convex ~~Robust~~ Optimization problem

Logo?



Logo?