Refined Theory of Packages*

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The fluid approximation for PEPA usually considers large populations of simple interacting sequential components characterised by small local state spaces. A natural question which arises is whether it is possible to extend this technique to composite processes with arbitrary large local state spaces. In [1] the authors were able to give a positive answer for a certain class of models. The current paper will enlarge this class.

1 Introduction

Fluid approximations for PEPA have been concerned with the aggregation of sequential components with relatively small state spaces. However, this approach turns out to be numerically inconvenient if the model under study has a large local state space. In [1] the authors studied a class of models with large state spaces which enjoy an efficient fluid approximation. The elements of this class are called *grouped PEPA models with packages*, where a package consists of arbitrarily many interacting composite processes, also denoted *subsystems*. Formally, a subsystem is a *grouped PEPA model* in the sense of [2], and can have a big local state space; that is, subsystems play the role of the aforementioned models with large local state spaces. The main result of [1] is that one can relate the fluid approximation of a package to the fluid approximation of one single subsystem of the package by means of a *simplification function*. We will introduce in this work a *refined* version of this function which will be able to simplify a broader class of grouped PEPA models with packages.

2 Theory of Packages

2.1 Conservative Single Level Theory

This subsection gives a concise overview of the already developed theory, cf. [1] for details. We begin with a useful definition which permits the renaming of labels in grouped PEPA models, cf. [2].

Definition 1. *Let G be a grouped PEPA model and $\mathscr{G}(G) = \{H_1,\dots,H_n\}$. For a set of labels $\{J_1,\dots,J_n\}$ the grouped PEPA model $G[H_1/J_1,\dots,H_n/J_n]$ (alternatively, $G[H_i/J_i \mid 1 \le i \le n]$) is obtained from G by replacing H_k with J_k, for $1 \le k \le n$.*

For instance, it holds that $H\{E\} \underset{L}{\bowtie} I\{E'\}[H/J, I/K] = J\{E\} \underset{L}{\bowtie} K\{E'\}$. The next definition introduces packages formally. Our goal is to simplify all packages by means of a simplification function.

Definition 2 (Grouped PEPA Model with Packages)**.** *The syntax of grouped PEPA models with packages is given by the grammar*

$$P ::= \underset{L^s}{\bowtie}^{N_s} G^s \mid P \underset{L}{\bowtie} P \mid P/L,$$

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where s ranges over all the occurrences of the term $\bowtie_{L^s}^{N_s} G^s$ *in a model.* L^s *is a synchronisation set,* N_s *is a natural, and* G^s *is a grouped PEPA model with* $\mathscr{G}(G^s) = \{Ⓢ^p \mid p \in Pos_s\}$*, where* Pos_s *identifies all labels within the* G^s*. The term* $\bowtie_{L^s}^{N_s} G^s$ *is denominated a* package*. Furthermore, we require that* $L^s = \emptyset$ *if* $N_s = 1$.

A grouped PEPA model with packages is interpreted as an ordinary grouped PEPA model according to the following definition.

Definition 3. *Let P be a grouped PEPA model with packages. Its interpretation as a grouped PEPA model is obtained by replacing each occurrence* $\bowtie_{L^s}^{N_s} G^s$ *with*

$$\begin{cases} G_1^s & , N_s = 1, \\ G_1^s \bowtie_{L^s} G_2^s \bowtie_{L^s} \cdots \bowtie_{L^s} G_{N_s}^s & , N_s > 1, \end{cases}$$

where $G_i^s := G^s[Ⓢ^p / Ⓢ_i^p \mid p \in Pos_s]$*, for all* $1 \le i \le N_s$.

For instance, consider the following PEPA model.

$$Busy \stackrel{def}{=} (use, r_1).Idle \qquad Idle \stackrel{def}{=} (reset, r_2).Busy$$
$$Run \stackrel{def}{=} (use, r_3).Wake \qquad Wake \stackrel{def}{=} (start, r_4).Run$$
$$Think \stackrel{def}{=} (think, r_5).Launch \qquad Launch \stackrel{def}{=} (start, r_6).Think$$

$$Sys := \underbrace{\big(Busy[N_P] \bowtie_{\{use\}} Run[N_T]\big)}_{\text{Machine 1}} \bowtie_{\emptyset} \underbrace{\big(Busy[N_P] \bowtie_{\{use\}} Run[N_T]\big)}_{\text{Machine 2}} \bowtie_{\{start\}} Think[N_U]$$

This models the cooperation of N_U users with a *package* of two machines, where each machine is made up of N_P CPUs and N_T threads. A corresponding PEPA model with packages could be then

$$\bowtie_{\emptyset}^{2} ①^1\{Busy[N_P]\} \bowtie_{\{use\}} ①^2\{Run[N_T]\} \bowtie_{\{start\}} \bowtie_{\emptyset}^{1} ②^1\{Think[N_U]\}, \tag{1}$$

where $\bowtie_{\emptyset}^{2} ①^1\{Busy[N_P]\} \bowtie_{\{use\}} ①^2\{Run[N_T]\}$ translates to

$$\big(①_1^1\{Busy[N_P]\} \bowtie_{\{use\}} ①_1^2\{Run[N_T]\}\big) \bowtie_{\emptyset} \big(①_2^1\{Busy[N_P]\} \bowtie_{\{use\}} ①_2^2\{Run[N_T]\}\big).$$

The next definitions describe the *simplifiable* PEPA models with packages.

Definition 4. *Let P be a PEPA model with packages. The inner cooperation set of P, denoted by* $\mathscr{L}(P)$*, is given by*

$$\mathscr{L}(P) := \begin{cases} L^s & , P = \bowtie_{L^s}^{N_s} G^s, \\ \mathscr{L}(P_0) \cup \mathscr{L}(P_1) & , P = P_0 \bowtie_{L} P_1, \\ \mathscr{L}(P_0) & , P = P_0 / L. \end{cases}$$

Definition 5 (Simplifiable PEPA Model with Packages)**.** *A PEPA model with packages P is said to be simplifiable if for every occurrence* $P_0 \bowtie_{L} P_1$ *in P it holds that* $\mathscr{L}(P_0 \bowtie_{L} P_1) \cap L = \emptyset$.

We can define now the simplification function considered in [1], hereafter called the *conservative simplification function*.

Definition 6 (Scaling Function). *Let G be a grouped PEPA model. The scaling of G with respect to $N \in \mathbb{N}$, denoted by $\mathscr{S}_0(G,N)$, is given by*

$$\mathscr{S}_0(G,N) := \begin{cases} H\{D[N]\} & ,G = H\{D\}, \\ \mathscr{S}_0(G_0,N) \bowtie_{L} \mathscr{S}_0(G_1,N) & ,\ G = G_0 \bowtie_{L} G_1, \\ \mathscr{S}_0(G_0,N)/L & ,\ G = G_0/L. \end{cases}$$

Definition 7 (Simplification Function). *Let P be a simplifiable PEPA model with packages and V a component-counting function of P. The simplification of (P,V) is denoted by $\big(\mathscr{S}_1(P), \mathscr{S}_2(V)\big)$, where*

$$\mathscr{S}_1(P) := \begin{cases} \mathscr{S}_0(G^s, N_s) & ,P = \bowtie_{L^s}^{N_s} G^s, \\ \mathscr{S}_1(P_0) \bowtie_{L} \mathscr{S}_1(P_1) & ,P = P_0 \bowtie_{L} P_1, \\ \mathscr{S}_1(P_0)/L & ,P = P_0/L. \end{cases}$$

$$\mathscr{S}_2(V)_{(\text{Ⓢ}^p,E)} := \sum_{i=1}^{N_s} V_{(\text{Ⓢ}^p_i,E)}, \qquad \textit{for all } \text{Ⓢ}^p \in \mathscr{G}(\mathscr{S}_1(P)) \textit{ and } E \in \mathscr{B}(\mathscr{S}_1(P), \text{Ⓢ}^p).$$

One can show that the fluid approximation of a *simplifiable* model P is related in an *exact* way to the fluid approximation of its simplification $\mathscr{S}_1(P)$. For instance, the example (1) is simplifiable into

$$\Big(\text{①}^1\{Busy[2N_P]\} \bowtie_{\{use\}} \text{①}^2\{Run[2N_T]\}\Big) \bowtie_{\{start\}} \text{②}^1\{Think[N_U]\}.$$

The relation between the fluid approximation V of (1) and the fluid approximation W of its simplification is then

$$V_{(\text{①}^1_i,Busy)} = \frac{1}{2}W_{(\text{①}^1,Busy)}, \qquad V_{(\text{①}^2_i,Run)} = \frac{1}{2}W_{(\text{①}^2,Run)}, \qquad V_{(\text{②}^1_1,Think)} = W_{(\text{②}^1,Think)},$$
$$V_{(\text{①}^1_i,Idle)} = \frac{1}{2}W_{(\text{①}^1,Idle)}, \qquad V_{(\text{①}^2_i,Wake)} = \frac{1}{2}W_{(\text{①}^2,Wake)}, \qquad V_{(\text{②}^1_1,Launch)} = W_{(\text{②}^1,Launch)},$$

where $1 \le i \le 2$. The next theorems establish the relation between the fluid approximations of the original and the simplified model in the general case. Their proofs may be found in [1].

Theorem 1. *Let Sys be a PEPA model with packages and V the unique solution of its underlying ODE system. Then we have that $V_{(\text{Ⓢ}^p_i,E)}(t) = V_{(\text{Ⓢ}^p_j,E)}(t)$ for all $\text{Ⓢ}^p_i, \text{Ⓢ}^p_j \in \mathscr{G}(Sys)$, $E \in \mathscr{B}(Sys, \text{Ⓢ}^p_i)$ and $t \ge 0$.*

Theorem 2. *For all simplifiable models with packages Sys we have that $\mathscr{S}_2(V, Sys)$ is the unique ODE solution of $\mathscr{S}_1(Sys)$ if V is the unique ODE solution of Sys.*

Let us consider now the following generalised version of our example (1) with N machines.

$$\Big(\bowtie_{\emptyset}^{N} \text{①}^1\{Busy[N_P]\} \bowtie_{\{use\}} \text{①}^2\{Run[N_T]\}\Big) \bowtie_{\{start\}} \bowtie_{\emptyset}^{1} \text{②}^1\{Think[N_U]\}.$$

The fluid approximation of this model is given then by an ODE system of size $4N+2$, whereas the fluid approximation of its simplification

$$\Big(\text{①}^1\{Busy[N \cdot N_P]\} \bowtie_{\{use\}} \text{①}^2\{Run[N \cdot N_T]\}\Big) \bowtie_{\{start\}} \text{②}^1\{Think[N_U]\}$$

is an ODE system of size $4+2$. That is, we can infer the solution of a problem which is not feasible in general (e.g. if N is too large) from the solution of a problem which is feasible.

2.2 Refined Single Level Theory

Let us define now a *refined* version of the simplification function.

Definition 8 (Refined Simplification Function)**.** *Let P be a grouped PEPA model with packages and V a component-counting function of P. The (refined) simplification of (P,V) is denoted by $\big(\mathscr{S}_1'(P), \mathscr{S}_2'(V,P)\big)$, where*

$$\mathscr{S}_1'(P,L) := \begin{cases} \mathscr{S}_1'(P_1, L \cup L_0) \underset{L_0}{\bowtie} \mathscr{S}_1'(P_2, L \cup L_0) & , P = P_1 \underset{L_0}{\bowtie} P_2 \\ \mathscr{S}_1'(P_1, L)/L_1 & , P = P_1/L_1 \\ \underset{L^s}{\bowtie}^{N_s} G^s & , P = \underset{L^s}{\bowtie}^{N_s} G^s \wedge L^s \cap L \neq \emptyset \\ \mathscr{S}_0(G^s, N_s) & , P = \underset{L^s}{\bowtie}^{N_s} G^s \wedge L^s \cap L = \emptyset \end{cases}$$

$$\mathscr{S}_2'(V,P,L)_{(X,E)} := \begin{cases} V_{(Ⓢ_i^p, E)} & , X = Ⓢ_i^p \\ \sum_{i=1}^{N_s} V_{(Ⓢ_i^p, E)} & , X = Ⓢ^p \end{cases} , \quad \textit{for all} \quad (X,E) \in \mathscr{B}(\mathscr{S}_1'(P,L))$$

and

$$\mathscr{S}_1'(P) := \mathscr{S}_1'(P,\emptyset) , \qquad \mathscr{S}_2'(V,P)_{(X,E)} := \mathscr{S}_2'(V,P,\emptyset)_{(X,E)} .$$

Informally, a package $\underset{L^s}{\bowtie}^{N_s} G^s$ will be simplifiable in a system *Sys*, if on the syntax tree path from the root (i.e., *Sys*) to the leaf $\underset{L^s}{\bowtie}^{N_s} G^s$ lie no shared action which is contained in L^s. One can prove then the following result.

Theorem 3. *For all models with packages Sys we have that $\mathscr{S}_2'(V, Sys)$ is the unique ODE solution of $\mathscr{S}_1'(Sys)$ if V is the unique ODE solution of Sys.*

Proof. See Appendix A. □

3 Concluding Remarks

We conclude the paper with a comparison of conservative and refined theory. The simplification $\mathscr{S}_1'(Sys)$ of *Sys* which is given by

$$Sys := \Big(\underset{\{start\}}{\bowtie}^{N_1} ①^1\{Busy[N_P]\} \underset{\{use\}}{\bowtie} ①^2\{Run[N_T]\}\Big) \underset{\{start\}}{\bowtie} \Big(\underset{\{reset\}}{\bowtie}^{N_2} ②^1\{Busy[N_P]\} \underset{\{use\}}{\bowtie} ②^2\{Run[N_T]\}\Big),$$

is then

$$\Big(\underset{\{start\}}{\bowtie}^{N_1} ①^1\{Busy[N_P]\} \underset{\{use\}}{\bowtie} ①^2\{Run[N_T]\}\Big) \underset{\{start\}}{\bowtie} \Big(②^1\{Busy[N_2 N_P]\} \underset{\{use\}}{\bowtie} ②^2\{Run[N_2 N_T]\}\Big),$$

i.e., we simplify the right, but not the left package. Theorems 1 and 3 imply then for the fluid approximation V of *Sys*

$$V_{(①_i^1, Busy)} = W_{(①_i^1, Busy)}, \quad V_{(①_i^1, Idle)} = W_{(①_i^1, Idle)}, \quad V_{(①_i^2, Run)} = W_{(①_i^2, Run)}, \quad V_{(①_i^2, Wake)} = W_{(①_i^2, Wake)},$$
$$V_{(②_j^1, Busy)} = \frac{1}{N_2} W_{(②^1, Busy)}, \quad V_{(②_j^1, Idle)} = \frac{1}{N_2} W_{(②^1, Idle)} \quad V_{(②_j^2, Run)} = \frac{1}{N_2} W_{(②^2, Run)}, \quad V_{(②_j^2, Wake)} = \frac{1}{N_2} W_{(②^2, Wake)},$$

if W denotes the fluid approximation of $\mathscr{S}_1'(Sys)$ and $1 \le i \le N_1$, $1 \le j \le N_2$. Since one can show that the refined simplification function will simplify all packages in a *simplifiable* model (cf. Definition 5) and *Sys* is not a *simplifiable* model, we infer that the refined theory is more flexible than the conservative one. In fact, one can observe from the proofs that the conservative theory gives sufficient and necessary conditions for simplification, as long as the simplification of *all* packages in a model is concerned. The refined theory instead simplifies all packages which can be simplified, i.e., any further package simplification (e.g., the simplification of the left package in the above system), would lead to a model with a fluid approximation which cannot be related to the fluid approximation of the original model. For instance, if U would denote the fluid approximation of

$$\Big(①^1\{Busy[N_1N_P]\} \underset{\{use\}}{\bowtie} ①^2\{Run[N_1N_T]\}\Big) \underset{\{start\}}{\bowtie} \Big(②^1\{Busy[N_2N_P]\} \underset{\{use\}}{\bowtie} ②^2\{Run[N_2N_T]\}\Big),$$

the relation

$$V_{(①^1_i,Busy)} = \frac{1}{N_1}U_{(①^1,Busy)}, \quad V_{(①^1_i,Idle)} = \frac{1}{N_1}U_{(①^1,Idle)}, \quad V_{(①^2_i,Run)} = \frac{1}{N_1}U_{(①^2,Run)}, \quad V_{(①^2_i,Wake)} = \frac{1}{N_1}U_{(①^2,Wake)},$$
$$V_{(②^1_j,Busy)} = \frac{1}{N_2}U_{(②^1,Busy)}, \quad V_{(②^1_j,Idle)} = \frac{1}{N_2}U_{(②^1,Idle)} \quad V_{(②^2_j,Run)} = \frac{1}{N_2}U_{(②^2,Run)}, \quad V_{(②^2_j,Wake)} = \frac{1}{N_2}U_{(②^2,Wake)},$$

where $1 \leq i \leq N_1$ and $1 \leq j \leq N_2$, would be in general wrong.

References

[1] M. Tschaikowski, M. Tribastone, Continuous-state explosion of fluid performance models, Submitted to Performance Evaluation.

[2] R. A. Hayden, J. T. Bradley, A fluid analysis framework for a markovian process algebra, Theor. Comput. Sci. 411 (22-24) (2010) 2260–2297.

A Proofs

Notation The symbol $\overset{A}{=}$ will be used in proofs to indicate an equality that follows from a statement A (e.g.: $\overset{L\,1}{=}$ denotes Lemma 1, $\overset{T\,1}{=}$ Theorem 1 and $\overset{I.H.}{=}$ the induction hypothesis).

Definition 9. *We let* $T^s_N = \underset{L^s}{\bowtie}{}^N G^s$*, for all* $1 \leq N \leq N_s$ *and* $T^s := T^s_{N_s}$.

Lemma 1. *Let* T^s *be a subprocess of a model with packages Sys and* V *the unique ODE solution of Sys. Then*

$$\forall ⓢ^p_i \in \mathscr{G}(T^s) \forall E \in \mathscr{B}(T^s, ⓢ^p_i) \forall \alpha \in \mathscr{A}^\tau \Big(\mathscr{R}_\alpha(T^s, V, ⓢ^p_i, E) = \mathscr{R}_\alpha(G^s_i, V, ⓢ^p_i, E)\Big)$$

Proof. Cf. [1]. □

Lemma 2. *Let* V *denote a component-counting function of a model with packages* P*. Then* $r_\alpha(P, uV) = ur_\alpha(P, V)$ *and* $\mathscr{R}_\alpha(P, uV, H, E) = u\mathscr{R}_\alpha(P, V, H, E)$ *for all* $H \in \mathscr{G}(P)$ *and* $E \in \mathscr{B}(P, H)$ *if* $(uV)(H, E) := uV(H, E)$ *for a real* $u > 0$.

Proof. Straightforward structural induction over P. □

Lemma 3. *Let* V *denote the ODE solution of a model with packages Sys. Then for all subprocesses* P *of Sys it holds*

$$\forall \alpha \in \mathscr{A} \forall L, \{\alpha\} \subseteq L \subseteq \mathscr{A} \Big(r_\alpha(P, V) = r_\alpha(\mathscr{S}'_1(P, L), \mathscr{S}'_2(V, P, L))\Big) .$$

Proof. We prove this by induction over the structure of P.

- $P = T^s$: since the case $L^s \cap L \neq \emptyset$ is clear, we assume $L^s \cap L = \emptyset$. This and $\alpha \in L$ imply then $\alpha \notin L^s$ which allows us to infer

$$r_\alpha(T^s, V) = \sum_{i=1}^{N_s} r_\alpha(G_i^s, V) \overset{\text{T 1}}{=} N_s r_\alpha(G_1^s, V) \overset{\text{L 2}}{=} r_\alpha(G_1^s, N_s V) \overset{\text{T 1}}{=} r_\alpha\Big(\mathscr{S}_1'(T^s, L), \mathscr{S}_2'(V, T^s, L)\Big).$$

- $P = P_1/L_1$: since the case $\alpha \in L_1$ is trivial we assume $\alpha \notin L_1$. Then

$$r_\alpha(P_1/L_1, V) = r_\alpha(P_1, V) \overset{\text{I.H.}}{=} r_\alpha\Big(\mathscr{S}_1'(P_1, L), \mathscr{S}_1'(V, P_1, L)\Big) = r_\alpha\Big(\mathscr{S}_1'(P_1/L_1, L), \mathscr{S}_1'(V, P_1/L_1, L)\Big).$$

- $P = P_1 \underset{L_0}{\bowtie} P_2$: then we get in the case $\alpha \in L_0$

$$\begin{aligned} r_\alpha\Big(\mathscr{S}_1'(P_1 \underset{L_0}{\bowtie} P_2, L), \mathscr{S}_2'(V, P_1 \underset{L_0}{\bowtie} P_2, L)\Big) &= \\ &= r_\alpha\Big(\mathscr{S}_1'(P_1, L \cup L_0) \underset{L_0}{\bowtie} \mathscr{S}_1'(P_2, L \cup L_0), \mathscr{S}_2'(V, P_1 \underset{L_0}{\bowtie} P_2, L)\Big) \\ &= \min\Big(r_\alpha(\mathscr{S}_1'(P_1, L \cup L_0), \mathscr{S}_2'(V, P_1, L \cup L_0)), r_\alpha(\mathscr{S}_1'(P_2, L \cup L_0), \mathscr{S}_2'(V, P_2, L \cup L_0))\Big) \\ &\overset{\text{I.H.}}{=} \min(r_\alpha(P_1, V), r_\alpha(P_2, V)) = r_\alpha(P_1 \underset{L_0}{\bowtie} P_2, V) \end{aligned}$$

and in the case $\alpha \notin L_0$

$$\begin{aligned} r_\alpha\Big(\mathscr{S}_1'(P_1 \underset{L_0}{\bowtie} P_2, L), \mathscr{S}_2'(V, P_1 \underset{L_0}{\bowtie} P_2, L)\Big) &= \\ &= r_\alpha\Big(\mathscr{S}_1'(P_1, L \cup L_0) \underset{L_0}{\bowtie} \mathscr{S}_1'(P_2, L \cup L_0), \mathscr{S}_2'(V, P_1 \underset{L_0}{\bowtie} P_2, L)\Big) \\ &= r_\alpha\Big(\mathscr{S}_1'(P_1, L \cup L_0), \mathscr{S}_2'(V, P_1, L \cup L_0)\Big) + r_\alpha\Big(\mathscr{S}_1'(P_2, L \cup L_0), \mathscr{S}_2'(V, P_2, L \cup L_0)\Big) \\ &\overset{\text{I.H.}}{=} r_\alpha(P_1, V) + r_\alpha(P_2, V) = r_\alpha(P_1 \underset{L_0}{\bowtie} P_2, V). \end{aligned}$$

□

Theorem 4. *Let V denote the ODE solution of a model with packages Sys. Then for all subprocesses P of Sys it holds*

$$\begin{aligned} X = ⓢ^p \quad &\Longrightarrow \quad \mathscr{R}_\alpha(\mathscr{S}_1'(P, L), \mathscr{S}_2'(V, P, L), ⓢ^p, E) = \sum_{i=1}^{N_s} \mathscr{R}_\alpha(P, V, ⓢ_i^p, E)\ , \\ X = ⓢ_i^p \quad &\Longrightarrow \quad \mathscr{R}_\alpha(\mathscr{S}_1'(P, L), \mathscr{S}_2'(V, P, L), ⓢ_i^p, E) = \mathscr{R}_\alpha(P, V, ⓢ_i^p, E)\ , \end{aligned}$$

for all $\alpha \in \mathscr{A}^\tau$, $L \subseteq \mathscr{A}$ and $(X, E) \in \mathscr{B}(\mathscr{S}_1'(P, L))$.

Proof. We prove this by induction over the structure of P.

- $P = T^s$: since the case $L^s \cap L \neq \emptyset$ is clear, we focus on the case $L \cap L^s = \emptyset$. Then $X = ⓢ^p$ and

$$\begin{aligned} \sum_{i=1}^{N_s} \mathscr{R}_\alpha(T^s, V, ⓢ_i^p, E) \overset{\text{L 1}}{=} \sum_{i=1}^{N_s} \mathscr{R}_\alpha(G_i^s, V, ⓢ_i^p, E) \overset{\text{T 1}}{=} N_s \mathscr{R}_\alpha(G_1^s, V, ⓢ_1^p, E) \\ \overset{\text{L 2}}{=} \mathscr{R}_\alpha(G_1^s, N_s V, ⓢ_1^p, E) \overset{\text{T 1}}{=} \mathscr{R}_\alpha(\mathscr{S}_1'(T^s, L), \mathscr{S}_2'(V, T^s, L), ⓢ^p, E) \end{aligned}$$

- $P = P_1 \bowtie_{L_0} P_2$: since $X \in \mathscr{G}(\mathscr{S}_1'(P,L)) = \mathscr{G}(\mathscr{S}_1'(P_1, L\cup L_0)) \cup \mathscr{G}(\mathscr{S}_1'(P_2, L\cup L_0))$ we may assume w.l.o.g. that $X \in \mathscr{G}(\mathscr{S}_1'(P_1, L\cup L_0))$. Let us consider first the case $\alpha \in L_0$. Then

$$\begin{aligned}
\mathscr{R}_\alpha\Big(\mathscr{S}_1'(P,L),\mathscr{S}_2'(V,P,L),X,E\Big) &= \frac{\mathscr{R}_\alpha\Big(\mathscr{S}_1'(P_1,L\cup L_0),\mathscr{S}_2'(V,P_1,L\cup L_0),X,E\Big)}{r_\alpha\Big(\mathscr{S}_1'(P_1,L\cup L_0),\mathscr{S}_2'(V,P_1,L\cup L_0)\Big)} \quad \times \\
&\times \quad \min\Big(r_\alpha\Big(\mathscr{S}_1'(P_1,L\cup L_0),\mathscr{S}_2'(V,P_1,L\cup L_0)\Big), r_\alpha\Big(\mathscr{S}_1'(P_2,L\cup L_0),\mathscr{S}_2'(V,P_2,L\cup L_0)\Big)\Big) \\
&\overset{\text{L 3}}{=} \frac{\mathscr{R}_\alpha\Big(\mathscr{S}_1'(P_1,L\cup L_0),\mathscr{S}_2'(V,P_1,L\cup L_0),X,E\Big)}{r_\alpha(P_1,V)} \min\big(r_\alpha(P_1,V), r_\alpha(P_2,V)\big) \\
&\overset{\text{I.H.}}{=} \begin{cases} \dfrac{\sum_{i=1}^{N_s}\mathscr{R}_\alpha\Big(P_1,V,Ⓢ_i^p,E\Big)}{r_\alpha(P_1,V)} \min\big(r_\alpha(P_1,V),r_\alpha(P_2,V)\big) = \displaystyle\sum_{i=1}^{N_s}\mathscr{R}_\alpha(P,V,Ⓢ_i^p,E) & ,\ X = Ⓢ^p \\ \dfrac{\mathscr{R}_\alpha\Big(P_1,V,Ⓢ_i^p,E\Big)}{r_\alpha(P_1,V)} \min\big(r_\alpha(P_1,V),r_\alpha(P_2,V)\big) = \mathscr{R}_\alpha(P,V,Ⓢ_i^p,E) & ,\ X = Ⓢ_i^p \end{cases}
\end{aligned}$$

Let us consider now the case $\alpha \notin L_0$. Then

$$\begin{aligned}
\mathscr{R}_\alpha\Big(\mathscr{S}_1'(P,L),\mathscr{S}_2'(V,P,L),X,E\Big) &= \mathscr{R}_\alpha\Big(\mathscr{S}_1'(P_1,L\cup L_0),\mathscr{S}_2'(V,P_1,L\cup L_0),X,E\Big) \\
&\overset{\text{I.H.}}{=} \begin{cases} \displaystyle\sum_{i=1}^{N_s}\mathscr{R}_\alpha(P_1,V,Ⓢ_i^p,E) = \sum_{i=1}^{N_s}\mathscr{R}_\alpha(P,V,Ⓢ_i^p,E) & ,\ X = Ⓢ^p \\ \mathscr{R}_\alpha(P_1,V,Ⓢ_i^p,E) = \mathscr{R}_\alpha(P,V,Ⓢ_i^p,E) & ,\ X = Ⓢ_i^p \end{cases}
\end{aligned}$$

- $P = P_1/L_1$: follows with I.H. and a case distinction on α and X.

□

Proof of Theorem 3. Let us fix $X \in \mathscr{G}(\mathscr{S}_1'(Sys))$ and $E \in \mathscr{B}(\mathscr{S}_1'(Sys), X)$. Then

$$\begin{aligned}
&\frac{d}{dt}\mathscr{S}_2'(V,Sys)_{(Ⓢ^p,E)}(t) = \sum_{i=1}^{N_s}\frac{d}{dt}V_{(Ⓢ_i^p,E)}(t) \\
&= \sum_{i=1}^{N_s}\sum_{\alpha\in\mathscr{A}^\tau}\Big(\Big(\sum_{E'\in\mathscr{B}(Sys,Ⓢ_i^p)} p_{t(\alpha)}(E',E)\mathscr{R}_\alpha(Sys,V(t),Ⓢ_i^p,E')\Big) - \mathscr{R}_\alpha(Sys,V(t),Ⓢ_i^p,E)\Big) \\
&= \sum_{\alpha\in\mathscr{A}^\tau}\Big(\Big(\sum_{E'\in\mathscr{B}(\mathscr{S}_1'(Sys),Ⓢ^p)} p_{t(\alpha)}(E',E)\sum_{i=1}^{N_s}\mathscr{R}_\alpha(Sys,V(t),Ⓢ_i^p,E')\Big) - \sum_{i=1}^{N_s}\mathscr{R}_\alpha(Sys,V(t),Ⓢ_i^p,E)\Big) \\
&\overset{\text{T 4}}{=} \sum_{\alpha\in\mathscr{A}^\tau}\Big(\Big(\sum_{E'\in\mathscr{B}(\mathscr{S}_1'(Sys),Ⓢ^p)} p_{t(\alpha)}(E',E)\mathscr{R}_\alpha(\mathscr{S}_1'(Sys),\mathscr{S}_2'(V,Sys)(t),Ⓢ^p,E')\Big) \\
&\qquad\qquad - \mathscr{R}_\alpha(\mathscr{S}_1'(Sys),\mathscr{S}_2'(V,Sys)(t),Ⓢ^p,E)\Big),
\end{aligned}$$

if $X = Ⓢ^p$. A similar calculation leads to the corresponding result for the case $X = Ⓢ_i^p$. Observing that $\mathscr{S}_1'(Sys)$ induces the initial value $\big\{\big((X,E),\mathscr{S}_2'(V,Sys)_{(X,E)}(0)\big) \mid (X,E) \in \mathscr{B}(\mathscr{S}_1'(Sys))\big\}$ yields then the claim. □