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Static versus dynamic longevity-risk hedging

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Abstract

This paper provides the static, swap-based hedge for an annuity, and compares it with the dynamic, delta-based hedge, achieved using longevity bonds. We assume that the longevity intensity is distributed according to a CIR-type process and provide closed-form derivatives prices and hedges, also in presence of an analogous CIR process for interest rate risk. Our calibration to 65-year old UK males shows that – once interest rate risk is perfectly hedged – the average hedging error of the dynamic hedge is moderate, and both its variance and the thickness of the tails of its distribution are decreasing with the rebalancing frequency. The spread over the basic "swap rate" which makes 99.5% quantile of the distribution of the dynamic hedging error equal to the cost of the static hedge lies between 0.01 and 0.04%.

Keywords: longevity risk, static vs. dynamic hedging, longevity swaps, longevity bonds.
JEL classification: G22, G32.

1 Introduction

Life insurance companies' portfolios are affected by so-called longevity risk, which is the risk that people live longer than expected when the company priced and reserved their policies. So, while increasing longevity is welcome from the social point of view, it is considered one of the risks that life insurance companies have to face. The ways in which at present they can cope with longevity is either by reinsuring it, as they used to do in the past, or by using more recent hedging approaches. Such strategies rely on the use of so-called mortality derivatives, which were first introduced by Blake and Burrows (2001). In the last decade the market for such instruments slowly developed and, while still lacking liquidity, allows insurers and pension funds to pursue these alternative de-risking strategies.

At present, actors seeking coverage against longevity risk can choose between a full, static hedge through a derivative, such as an s-forward or a longevity swap, or a partial, dynamic hedge. In the first case the whole excess of longevity is transferred to a third party, once and for all, and the coverage is not changed over time. In the second case, coverage is partial, often done through customized

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derivatives, which we will call longevity bonds. The partial nature of coverage calls for adjustment over time.

In this paper longevity risk is represented through the so-called stochastic longevity, i.e. by an intensity of mortality arrival which is itself a stochastic process. In order to keep the model tractable and to provide easy to implement hedges, we work in continuous time. In order to ensure positivity of the intensity and to have a longevity model which nicely couples with the modeling of interest rates, we assume that longevity itself follows a Feller, or Cox et al. (1985)-type (CIR) process.

Some previous works have focused on static hedging tools. Ngai and Sherris (2011), in particular, compared the effectiveness of static hedging through various derivatives. Rather few previous studies focus instead on dynamic hedging. Among them, Dahl et al. (2011) analyzed dynamic hedging via longevity swaps, analyzing the different performance of a constant and of a rebalanced strategy.

The original contribution of our paper lies in providing both static and dynamic, closed - form hedges for the CIR longevity process. The static hedge entails the use of a longevity swap, while the dynamic hedge is performed applying Delta-Gamma hedging strategy, as proposed by Luciano et al. (2012b). We couple the theoretical contribution with a calibrated example, and we compare the efficiency of the static versus the dynamic hedge. We determine the cost of the static hedge which would "equate", in a sense that we specify below, the hedging error of the partial coverage. We explore sensitivity with respect to different assumptions on the rebalancing frequency of the strategy, which is expected to affect the quality of hedging. We leave the study of the role of transaction costs and basis risk for further research.

The paper unfolds as follows: in Section 2 we set up the model for longevity and financial risk evaluation, in Section 3 we describe the liabilities to be hedged, in Section 4 we explain the static and dynamic hedging strategy, in Section 5 we compare their effectiveness on a calibrated model. The last Section summarizes and outlines further research.

2 Longevity and interest rate risk modelling

In order to model longevity and interest rate risk, we assume that mortality for a specific generation occurs according to a Poisson process, whose intensity is stochastic. We consider a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, which satisfies the usual assumptions, and on which a filtration $\mathcal{F}_t$ is defined. The measure $\mathbb{Q}$ is already the so-called risk-neutral measure. We will discuss below the relationship between this measure and the effective one.

We let the mortality intensity of a specific generation be described by a so-called Cox-Ingersoll and Ross (CIR) process, which is actually a Feller process, of the type:

$$d\lambda(t) = (a + b\lambda(t))dt + \sigma\sqrt{\lambda(t)}dW(t), \tag{1}$$

with $a > 0, b > 0, \sigma > 0$, $\lambda(0) = \lambda_0 \in \mathbb{R}^{++}$. The reason behind the assumption $b > 0$ is that the process is expected to have no mean reversion. The previous SDE describes the evolution (for a given generation) of the intensity of mortality arrival over calendar time. Because the generation ages over time, the previous drift simply tells that the expected change in intensity is affine and increasing with the intensity itself.

If the initial point λ_0 is strictly positive and the coefficients satisfy the following condition:

$$a \geq \frac{\sigma^2}{2}, \tag{2}$$

then the mortality intensity $\lambda(t)$ will be strictly positive for every t, almost surely. Hence, in order to obtain a satisfactory calibrated model for the intensity process, we impose this condition on the parameters during the calibration.

Consistently, we assume that the spot interest rate - or interest rate intensity - follows a CIR process of the type:

$$dr(t) = (\bar{a} - \bar{b}r(t))dt + \bar{\sigma}\sqrt{r(t)}dW'(t), \tag{3}$$

with $\bar{a} > 0, \bar{b} > 0, \bar{\sigma} > 0, r(0) = r_0 \in \mathbb{R}^{++}$, where the Wiener process W' is independent of W.[1] The last assumption entails independency between the whole longevity and interest intensity processes. The negative sign preceding $\bar{b}$ and its strict positivity guarantee that the process for the interest-rate incorporates mean reversion, which is a usual assumption in the interest-rate domain. The coefficient $\bar{b}$ is called speed of mean reversion and represents the speed at which the the short rate $r(t)$ returns to its long-run value $\bar{a}$ whenever $r < \bar{a}$ or viceversa.

Similarly to the longevity case, the restriction on the parameters that, together with the positivity of the initial point r_0, guarantees that the interest rate $r(t)$ never turns negative is given by:

$$\bar{a} \geq \frac{\bar{\sigma}^2}{2}. \tag{4}$$

At each single point in time, the conditional distributions of the mortality intensity and the interest rate are given, up to a scale factor, by a noncentral chi-square distribution. In details, given two time instants $u < t$, then the distribution of $\lambda(t)$ conditional on $\lambda(u)$ is given by:

$$\lambda(t) \approx \frac{\sigma^2\left(e^{b(t-u)} - 1\right)}{4b}\mathcal{X}_d'^2(\nu), \tag{5}$$

where $\mathcal{X}_d'^2(\nu)$ denotes the density of a noncentral chi-square random variable with degrees of freedom

$$d = \frac{4a}{\sigma^2}, \tag{6}$$

and noncentrality parameter

$$\nu = \frac{4be^{b(t-u)}}{\sigma^2\left(e^{b(t-u)} - 1\right)}\lambda(u). \tag{7}$$

Similarly, the distribution of $r(t)$ conditional on $r(u)$ is given by:

$$r(t) \approx \frac{\bar{\sigma}^2\left(1 - e^{-\bar{b}(t-u)}\right)}{4\bar{b}}\mathcal{X}_{\bar{d}}'^2(\bar{\nu}), \tag{8}$$

[1] Let the filtration $\mathcal{F}_t$ be the filtration generated by the two Brownian motions.

where $\mathcal{X}_{\bar{d}}^{\prime 2}(\bar{\nu})$ denotes the density of a noncentral chi-square random variable with degrees of freedom

$$\bar{d} = \frac{4\bar{a}}{\bar{\sigma}^2}, \tag{9}$$

and noncentrality parameter

$$\bar{\nu} = \frac{4\bar{b}e^{-\bar{b}(t-u)}}{\bar{\sigma}^2(1 - e^{-\bar{b}(t-u)})} r(u). \tag{10}$$

In order to proceed to insurance products pricing and hedging, the risk-neutral dynamics of the two previous processes is needed. However, for calibration purposes, its effective or historical version may be useful, at least for the longevity case. In order to keep the notation simple, we just assume that there is no risk premium in the longevity market or, equivalently, that equation (1) holds under both measures. Therefore, the calibration of the longevity intensity is performed by estimating its dynamics under the historical measure and, then, using it also under the risk-neutral measure. The calibration of the interest rate dynamics is, on the other hand, performed directly under the risk-neutral measure, thus incorporating the risk premium.

If we call τ the time to death, the conditional survival probability from t to T is

$$S(t,T) = \mathbb{P}\left(\tau \geqslant T \mid \tau > t\right).$$

where $\mathbb{P}$ is in the $Q-$measure. In the presence of a stochastic intensity, it can be represented as

$$S(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right) \mid \mathcal{F}_t\right]. \tag{11}$$

The expectation $\mathbb{E}$, here and below, is still under $\mathbb{Q}$. Under the CIR assumption, that probability becomes:

$$S(t,T) = A(t,T)e^{-B(t,T)\lambda(t)}, \tag{12}$$

where $A(t,T)$ and $B(t,T)$ are solutions of an appropriate system of Riccati equations. These functions are

$$A(t,T) = \left(\frac{2\gamma e^{\frac{1}{2}(\gamma-b)(T-t)}}{(\gamma-b)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right)^{\frac{2a}{\sigma^2}}, \tag{13}$$

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma-b)\left(e^{\gamma(T-t)}-1\right)+2\gamma}, \tag{14}$$

where $\gamma = \sqrt{b^2 + 2\sigma^2}$. As shown in Fung et al. (2014), the above specification guarantees also that the limit of the survival probability, when T diverges, is zero.

For any given t, it is possible to compute the log derivative of the survival probability, which is somewhat inappropriately called the " forward" mortality intensity for time T, since it represents its forecast at time t. By definition

$$f(t,T) = -\frac{\partial lnS(t,T)}{\partial T} = -\frac{\partial lnA(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T}\lambda(t), \tag{15}$$

where

$$\frac{\partial lnA(t,T)}{\partial T} = \frac{2a}{\sigma^2}\left[\frac{1}{2}(\gamma - b) - \frac{\gamma e^{\gamma(T-t)}}{e^{\gamma(T-t)} - 1 + \frac{2\gamma}{\gamma - b}}\right], \quad (16)$$

$$\frac{\partial B(t,T)}{\partial T} = \frac{4\gamma^2 e^{\gamma(T-t)}}{\left[(\gamma - b)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma\right]^2}. \quad (17)$$

Using a technique described in Jarrow and Turnbull (1994) and Luciano et al. (2012a), which exploits the definition of "forward" intensity, we can write the survival as

$$S(t,T) = e^{-X(t,T)I(t)+Y(t,T)}, \quad (18)$$

where $I(t) = \lambda(t) - f(0,t)$, while $X(t,T)$ and $Y(t,T)$ are deterministic functions of parameters and time t and T:

$$X(t,T) = B(t,T),$$

$$Y(t,T) = lnA(t,T) - B(t,T)\left[-\frac{\partial lnA(0,t)}{\partial t} + \frac{\partial B(0,t)}{\partial t}\lambda(0)\right].$$

The term I is called *longevity risk factor* and is the difference between the actual and forecasted intensity for time t. Using the fact that $\lambda(t) = I(t) + f(0,t)$, (18) becomes

$$S(t,T) = A(t,T)e^{-B(t,T)[I(t) - \frac{\partial lnA(0,t)}{\partial t} + \frac{\partial B(0,t)}{\partial t}\lambda(0)]}. \quad (19)$$

Hence, we have an expression for the survival equivalent to (12). This expression will play a crucial role in hedging, because it encapsulates all riskiness in the I factor, which has the intuitively nice interpretation of difference between the forecasted and actual intensity. This is exactly what we have in mind when we think of *longevity risk*.

The discount factor or bond price for time t, under any stochastic process for the spot rate, is

$$D(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{F}_t\right], \quad (20)$$

which, in the CIR case, becomes

$$D(t,T) = \bar{A}(t,T)e^{-\bar{B}(t,T)r(t)},$$

$$\bar{A}(t,T) = \left(\frac{2\bar{\gamma}e^{\frac{1}{2}(\bar{\gamma}+\bar{b})(T-t)}}{(\bar{\gamma}+\bar{b})\left(e^{\bar{\gamma}(T-t)} - 1\right) + 2\bar{\gamma}}\right)^{\frac{2\bar{a}}{\bar{\sigma}^2}}, \quad (21)$$

$$\bar{B}(t,T) = \frac{2\left(e^{\bar{\gamma}(T-t)} - 1\right)}{(\bar{\gamma}+\bar{b})\left(e^{\bar{\gamma}(T-t)} - 1\right) + 2\bar{\gamma}}, \quad (22)$$

with $\bar{\gamma} = \sqrt{\bar{b}^2 + 2\bar{\sigma}^2}$. As in the longevity case, the bond value can be reformulated as

$$D(t,T) = e^{-\bar{X}(t,T)K(t)+\bar{Y}(t,T)}, \quad (23)$$

where

$$\begin{aligned}
\bar{X}(t,T) &= \bar{B}(t,T), \\
\bar{Y}(t,T) &= ln\bar{A}(t,T) - \bar{B}(t,T)\left[-\frac{\partial ln\bar{A}(0,t)}{\partial t} + \frac{\partial \bar{B}(0,t)}{\partial t} r(0)\right],
\end{aligned}$$

and K is the *financial risk factor*, measured by the difference between the short and forward rate:

$$K(t) = r(t) - F(0,t).$$

The forward rate $F(0,t)$ is a significant financial quantity, that represents the fair price at time 0 - and in general at time t when it becomes $F(t,T)$ - for a forward contract on the spot rate at T. It is computed, similarly to the forward mortality intensity, as

$$F(t,T) = -\frac{\partial \ln D(t,T)}{\partial T} = -\frac{\partial ln\bar{A}(t,T)}{\partial T} + \frac{\partial \bar{B}(t,T)}{\partial T} r(t), \tag{24}$$

where

$$\frac{\partial ln\bar{A}(t,T)}{\partial T} = \frac{2\bar{a}}{\bar{\sigma}^2}\left[\frac{1}{2}(\bar{\gamma}+\bar{b}) - \frac{\bar{\gamma}e^{\bar{\gamma}(T-t)}}{e^{\bar{\gamma}(T-t)} - 1 + \frac{2\bar{\gamma}}{\bar{\gamma}+\bar{b}}}\right], \tag{25}$$

$$\frac{\partial \bar{B}(t,T)}{\partial T} = \frac{4\gamma^2 e^{\bar{\gamma}(T-t)}}{\left[(\bar{\gamma}+\bar{b})\left(e^{\bar{\gamma}(T-t)} - 1\right) + 2\bar{\gamma}\right]^2}, \tag{26}$$

So, also in the bond case, the reformulation in terms of the risk factor allows us to synthetize in a unique spread the forecast error that economic agents can make and that they may be willing to hedge.

3 The insurance company portfolio

Let us suppose that the insurance company has sold a number n of annuities on the generation x whose mortality intensity is λ. In principle, its portfolio is likely to include also term insurance contracts, pure endowments or more complex products, but, for the purpose of our discussion, it seems sufficient to concentrate on annuities. The extension to the other contracts just listed is quite straightforward.

If liabilities are evaluated at fair value, an annuity - with annual installments R, paid at year-end - issued at time 0 to an individual belonging to generation x, lasting up to T and (already) paid through a single premium at policy inception is worth

$$N(t,T) = R\sum_{u=1}^{T-t} D(t,t+u)S(t,t+u). \tag{27}$$

Assuming a CIR mortality intensity (1) and a CIR interest rate process (3), we have that

$$N(t,T) = R\sum_{u=1}^{T-t} e^{-\bar{X}(t,t+u)K(t)+\bar{Y}(t,t+u)} \cdot e^{-X(t,t+u)I(t)+Y(t,t+u)}. \tag{28}$$

This is the so-called fair value of the reserves that the insurance company should have in order to face the payments for the generation under exam. Under the previous assumptions, if there is any unexpected change in the mortality intensity or the interest rate process, the marginal effect on the reserve is as follows:

$$dN = \frac{\partial N}{\partial I} dI + \frac{1}{2} \frac{\partial^2 N}{\partial I^2} (dI)^2 + \frac{\partial N}{\partial K} dK + \frac{1}{2} \frac{\partial^2 N}{\partial K^2} (dK)^2 ,$$

where

$$\frac{\partial N}{\partial I} = R \sum_{u=1}^{T-t} D(t, t+u) \Delta^M (t, t+u),$$

$$\frac{\partial^2 N}{\partial I^2} = R \sum_{u=1}^{T-t} D(t, t+u) \Gamma^M (t, t+u),$$

$$\frac{\partial N}{\partial K} = R \sum_{u=1}^{T-t} \Delta^F (t, t+u) S(t, t+u),$$

$$\frac{\partial^2 N}{\partial K^2} = R \sum_{u=1}^{T-t} \Gamma^F (t, t+u) S(t, t+u),$$

and the greeks against mortality and interest rate risk, denoted as $\Delta^M, \Gamma^M, \Delta^F, \Gamma^F$, are defined starting from (19) and (23). Appropriate derivations lead to

$$\frac{\partial S}{\partial I} = \Delta^M (t, T) = -X(t, T) S(t, T) \leq 0, \tag{29}$$

$$\frac{\partial^2 S}{\partial I^2} = \Gamma^M (t, T) = X(t, T)^2 S(t, T) \geq 0. \tag{30}$$

Analogously, the greeks for interest-rate risk are

$$\frac{\partial D}{\partial I} = \Delta^F (t, T) = -\bar{X}(t, T) D(t, T) \leq 0, \tag{31}$$

$$\frac{\partial^2 D}{\partial I^2} = \Gamma^F (t, T) = \bar{X}(t, T)^2 D(t, T) \geq 0. \tag{32}$$

4 Hedging Strategies: implementation

We discuss separately the static and the dynamic hedge. In order to hedge the unexpected changes just formalized, the insurance company can either buy a static hedge, i.e. a derivative, or set up an approximated, partial hedge, that can then be revised over time.

4.1 Static hedge

For longevity, the static hedge can be provided by a so-called s-swap or longevity swap. A longevity swap is a sequence of s-forwards.

An s-forward signed at t is a contract in which one party agrees to pay a fixed amount in exchange for the number of survivors belonging to a specific generation x in a given time period. We normalize the number of individuals in generation x to one. We thus abstract from idiosyncratic risk and consider a single annuity as equivalent to a well-diversified homogeneous portfolio of annuities. If the maturity of the forward is T, and the fixed payment is $K(T)$, then the payoff at maturity, from the point of view of who pays fixed, is

$$\exp\left(-\int_t^T \lambda_x(s)ds\right) - K(T), \tag{33}$$

where λ_x is the mortality intensity of generation x. An s-forward (unit hedge) helps providers of annuities to hedge their exposure: if the provider sold a pure endowment on generation x with maturity T, and buys an s-forward, he will pay $K(T)$ for sure instead of being exposed to the randomness of the payment $\exp\left(-\int_t^T \lambda_x(s)ds\right)$. Under the assumption of no arbitrage, and assuming independence between mortality and interest-rate risk, the fair value at time t of such a contract is

$$\begin{aligned} & [S(t,T) - K(T)]\, D(t,T) = \\ = \quad & \mathbb{E}_t\left[\exp\left(-\int_t^T \lambda_x(s)ds\right) - K(T)\right] \mathbb{E}_t\left[\exp\left(-\int_t^T r(u)du\right)\right]. \end{aligned}$$

where the index t signals that the expectation is the $\mathcal{F}_t-$ one. Since, in order to enter such a contract, no price is paid at inception, the no-arbitrage value of $K(T)$, which equates the fair value to zero, is $S(t,T)$.

A longevity swap is a sequence of s-forwards. If the exchange of amounts happens once a year, the payment for the period $(T-1,T)$ is $K(T)$ and the contract lasts until the last individual of the generation is dead (at age ω), the payoffs are given by (33) for $T = 1,..,\omega - t$.

Under the assumption of no arbitrage, and still assuming independence between mortality and interest-rate risk, the value at time t of such a contract is

$$\begin{aligned} & \sum_{T=t+1}^{\omega-t} [S(t,T) - K(T)]\, D(t,T) = \\ & = \sum_{T=t+1}^{\omega-t} \mathbb{E}_t\left[\exp\left(-\int_t^T \lambda_x(s)ds\right) - K(T)\right] \mathbb{E}_t\left[\exp\left(-\int_t^T r(u)du\right)\right], \end{aligned}$$

which is equal to zero, as a fair pricing would require, if $K(T)$ is set equal to the survival probability for time T. We call $K(T)$ the swap rate for the time period $(T-1,T)$.[2]

Usually the previous swap is not offered to the insurance company at fair value. It entails a cost, which we take to be fixed and equal to C_0. It follows

[2] An alternative would be to fix a unique swap rate for all periods, $K(T) = K$. In this case fairness would be guaranteed by setting K equal to the following value:

$$K = \frac{\sum_{T=t+1}^{\omega-t} \mathbb{E}\left[\exp\left(-\int_t^T \lambda_x(s)ds\right)\right] \mathbb{E}\left[-\exp\left(\int_t^T r(u)du\right)\right]}{\sum_{T=1}^{\omega-t} \mathbb{E}\left[-\exp\left(\int_t^T r(u)du\right)\right]}.$$

that the fees $K(T)$ are raised to $K'(T)$, where the sequence $K'(T)$ solves

$$-C_0 = \sum_{T=t+1}^{\omega-t} \left[S(t,T) - K'(T) \right] D(t,T).$$

For the sake of simplicity, we assume that the cost C_0 is evenly distributed along the "life" of the swap, by increasing the swap rates K by the same amount, i.e. $K'(T) = K(T)(1+m) = S(t,T)(1+m)$ where m is determined as follows:

$$\begin{aligned} -C_0 &= \sum_{T=t+1}^{\omega-t} \left[S(t,T) - K(T)(1+m) \right] D(t,T) \\ &= -m \sum_{T=t+1}^{\omega-t} S(t,T) D(t,T), \end{aligned}$$

which implies that

$$\begin{aligned} m &= \frac{C_0}{\sum\limits_{T=t+1}^{\omega-t} S(t,T) D(t,T)} \\ &= \frac{C_0}{\sum\limits_{T=t+1}^{\omega-t} e^{-\bar{X}(t,T)K(t)+\bar{Y}(t,T)} \cdot e^{-X(t,T)I(t)+Y(t,T)}}. \end{aligned}$$

In principle, the insurance company can be interested in hedging interest rate risk too. We neglect this coverage here. However, given the similarity of the two processes, the formulas for an interest rate swap would be similar to the survival one.

4.2 Dynamic hedge

An alternative to the previous hedge is the following: cover only the changes in the fair value of the reserve (the liabilities) approximated at the first or second order. This is known as delta, or delta-gamma, hedging (see Luciano et al. (2012a)). Both the first and second-order changes in the reserve, delta and gamma, depend on changes of the CIR longevity intensity and their expression has been already given explicitly in Section 3. For consistency with the static hedge, we assume that interest rate risk is not covered. For the same reason, we also assume that the dynamic hedge is self-financing, a requirement formalized below.

In order to cover the annuity against the two changes, a possibility is that of setting up a portfolio comprehensive of longevity bonds. Our longevity bonds pay at every year-end the survivorship of the reference generation.[3] Their payoff for year T is then

$$\exp\left(-\int_t^T \lambda_x(s) ds \right).$$

[3] If there is no longevity bond for a specific generation, basis risk arises: see for instance Cairns, Blake, Dowd, and MacMinn (2006).

Under no-arbitrage, if the bond maturity is T_i, its fair value at time t, $M_i(t)$, is

$$M_i(t) = S(t,T_i)D(t,T_i),$$

which, using the CIR assumption, can be written as

$$M_i(t) = \bar{A}(t,T_i)e^{-\bar{B}(t,T_i)r(t)}A(t,T_i)e^{-B(t,T_i)\lambda_x(t)},$$

or, in the Jarrow and Turnbull formulation, as

$$M_i(t) = e^{-\bar{X}(t,T_i)K(t)+\bar{Y}(t,T_i)}e^{-X(t,T_i)I(t)+Y(t,T_i)}.$$

In order to delta-gamma hedge and keep the hedge self financing, we need at each point in time three bonds, whose maturity is kept constant along the life of the hedge. The three bonds have maturities $T_i, i = 1,2,3$ and the number of bonds in the portfolio is $n_i, i = 1,2,3$.

At each rebalancing point t, the amount of the bonds used to hedge can be found by solving the following system

$$\begin{aligned}
-n\frac{\partial N(t)}{\partial I}dI + \sum_{i=1}^{3} n_i \frac{\partial M_i(t)}{\partial I}dI &= 0, \\
-n\frac{\partial^2 N(t)}{\partial I^2}(dI)^2 + \sum_{i=1}^{3} n_i \frac{\partial^2 M_i(t)}{\partial I^2}(dI)^2 &= 0, \\
-nN + \sum_{i=1}^{3} n_i M_i(t) &= 0.
\end{aligned} \tag{34}$$

The first equation nullifies the delta of the portfolio, the second nullifies the gamma, while the third requires it to be self-financing. Notice that the terms associated to the annuity enter with negative signs, as they represent the liability that the company is endowed with. Note also that the longevity bond value is equal to an annuity with a unique cash flow, or a pure endowment. The difference, from the standpoint of an insurance company, is that it can sell annuities and pure endowments – or reduce its exposure through reinsurance – and buy longevity bonds, while, at least in principle, it cannot do the converse.[4] We could use a number of other instruments to cover the annuity, starting from life assurances or death bonds, which pay the benefit in case of death. We restrict the attention to longevity bonds for the sake of simplicity. Let us also recall that longevity bonds – together with the life assurance and death bonds – represent the Arrow-Debreu securities of the insurance market. Once hedging is provided for them, it can be extended to every more complicated instrument. Immediately before each rebalancing date t we evaluate the portfolio. Its value is the gain or loss of the hedging strategy, which we finance through the bank account. In other words, at each rebalancing date we sell the entire portfolio and re-apply the self-financing delta-gamma strategy using the same instruments and solving equations (34). Any gain or loss from the hedging revision is stored or charged in the bank account, from which the payments due because of the annuity contract are also taken. The bank account accrues or charges the short interest rate $r(t)$. We refer to the absolute value of the bank account as to the hedging error.

[4] Reinsurance companies have less constraints in this respect. For instance, they can swap pure endowments or issue longevity bonds: see for instance Cowley and Cummins (2005).

5 Hedging Strategies: effectiveness and performance comparison

In order to compare the two strategies (static and dynamic) above, here we proceed as follows: we calibrate the models to the observed mortality rates of 65-year old UK males, we imagine different revision frequencies of the dynamic strategy, and then determine the cost of the static hedge which would equate the hedging error of the partial coverage, under different assumptions on the rebalancing frequency of the second. We focus on longevity risk hedging, thus assuming that interest-rate risk has already been hedged perfectly.

5.1 Calibration

We calibrate the parameters of our mortality model on the generation of UK males born in 1946, who were aged 65 on 31/12/2010 (i.e. $x = 65$). We fit our model minimizing the Rooted Mean Squared Error (RMSE) between the model-implied and the observed survival probabilities as computed from data provided by the Human Mortality Database. Under the constraint given by condition (2), we fix 01/01/1991 as the observation point (individuals have all reached aged 44) and we fit the observed survival probabilities $S(0,t)$ with t=1,...20. We collect the parameters and the calibration error in Table 1.

Table 1. Mortality Intensity Calibration results.

a	b	σ	Calibration Error
$4.13 \cdot 10^{-5}$	0.0709	0.0087	0.00006

Because condition (2) holds, the simulated mortality intensities $\lambda(t)$ will be strictly positive. In the simulations, we assume that the maximum life-span of an individual belonging to generation x is $\omega = 115$, hence the time horizon we use for the simulations of the intensity process is 50 years. Some simulated sample paths of the $\lambda_x(t)$ process are shown in figure 1.

Having abstracted from hedging issues concerning interest rate risk, we set the interest rate to a constant value $r = 0.02$.

5.2 Rebalancing frequency and dynamic hedging performance

In this section we compute the performance of the dynamic hedging strategy we described in Section 4.2 under different rebalancing frequencies and use the results to assess reasonable ranges for the cost of a longevity swap, as described in Section 4.1. Let us consider an annuity provider who has sold a whole-life annuity written on UK males aged 65 at time 0 (maturity $T_A = 50y$). We assume moreover the existence of three longevity bonds with rolling maturities 10, 15 and 20 years, written on the same generation of 65-year-old UK males. Figure 2 provides simulated sample paths for the value of those bonds.
Next, we need to decide after how many years we want to assess the perfomance

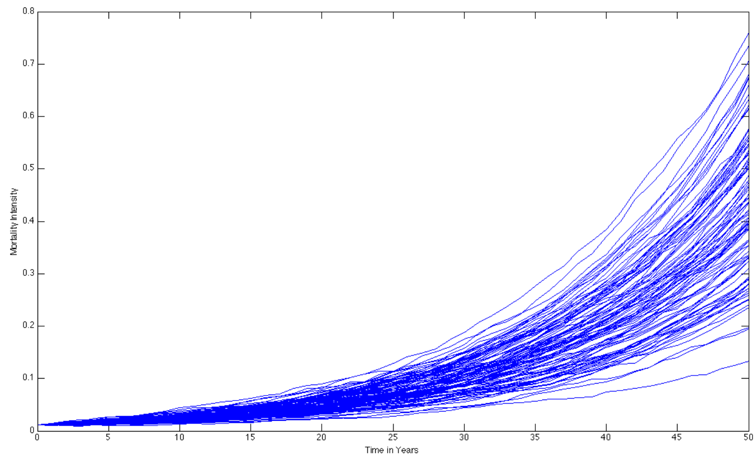

Figure 1. Simulated sample paths of the mortality intensity process $\lambda_x(t)$.

of the hedge. In principle, since the value of the annuity is computed taking into account the maximum life-span $\omega = 115$ of an individual belonging to generation x, then, the implementation of the dynamic delta-gamma hedging strategy is of interest up to 50 years. By so doing, it does not disregard the tails of the distribution of deaths among policyholders. This procedure is justified since, when an insurance company decides to implement a longevity risk hedging strategy, she wants to protect herself against the risk that the realizations of death arrivals among her portfolio of annuitants might belong to the right-tail of the death distribution. However, given the initial age and the behaviour of survival probabilities, it is reasonable to assume that 30 years after the inception of the Annuity contract, the bulk of her initial obligations would be gone. This is why, in order to evaluate the effectiveness of the self-financing dynamic Delta-Gamma hedging strategy, we fix a time horizon of 30 years. We consider three different rebalancing frequencies of 3 months, 6 months and 1 year, respectively. Sample paths of the simulated evolution of the annuity value, using these three different step sizes, are represented in Figure 3.

Simulated samples of the evolution of the Bank Account, for each rebalancing frequency, are given in Figure 4. The variability of each path is larger at the beginning of the horizon, when the value of the annuity is higher, and decreases with time. Figure 5 reports the distribution of the value of the bank account at our horizon of interest, t=30 years, for the three different rebalancing frequencies. The picture shows that the average cost of the hedging strategy is higher the longer the time interval between two revisions of the strategy. Also, increasing the rebalancing frequency reduces remarkably the dispersion of the value around its mean. The strategy rebalanced at 1-year frequency (solid line) presents the fattest tails. Table 2 contains, for each case, the mean and standard deviation of the hedging error after 30 years and allows to appreciate the effects of different rebalancing frequencies. Less frequent rebalancing leads to higher average hedging errors and higher variability, as expected. However, we remark that this result is obtained in the absence of transaction costs, which

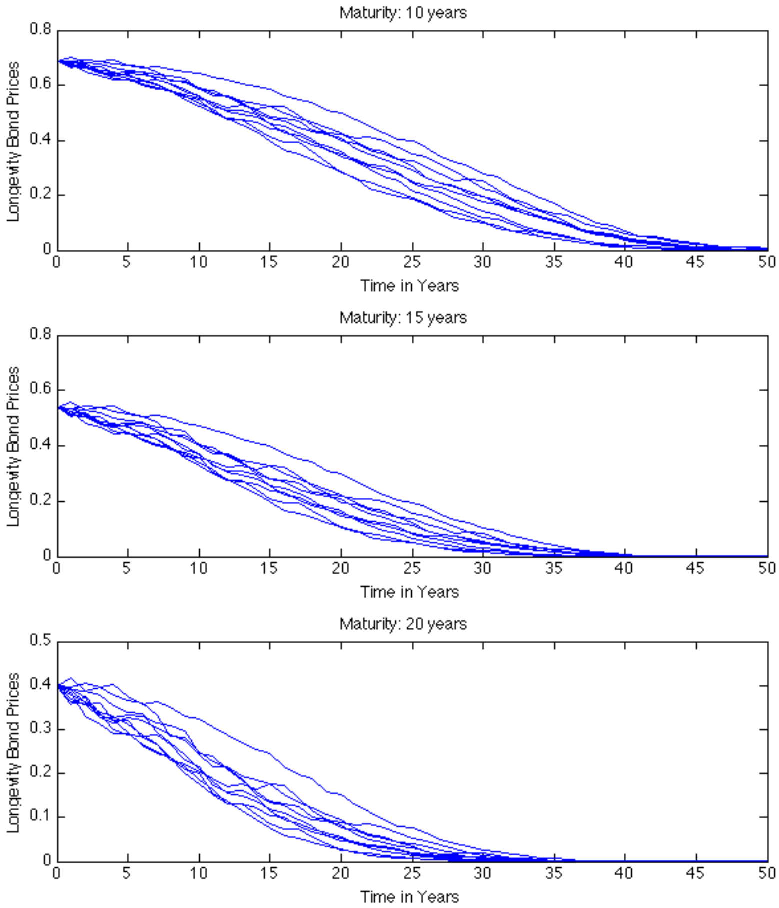

Figure 2. Simulated sample paths of the Longevity Bonds $M_i(t)$ written on generation x.

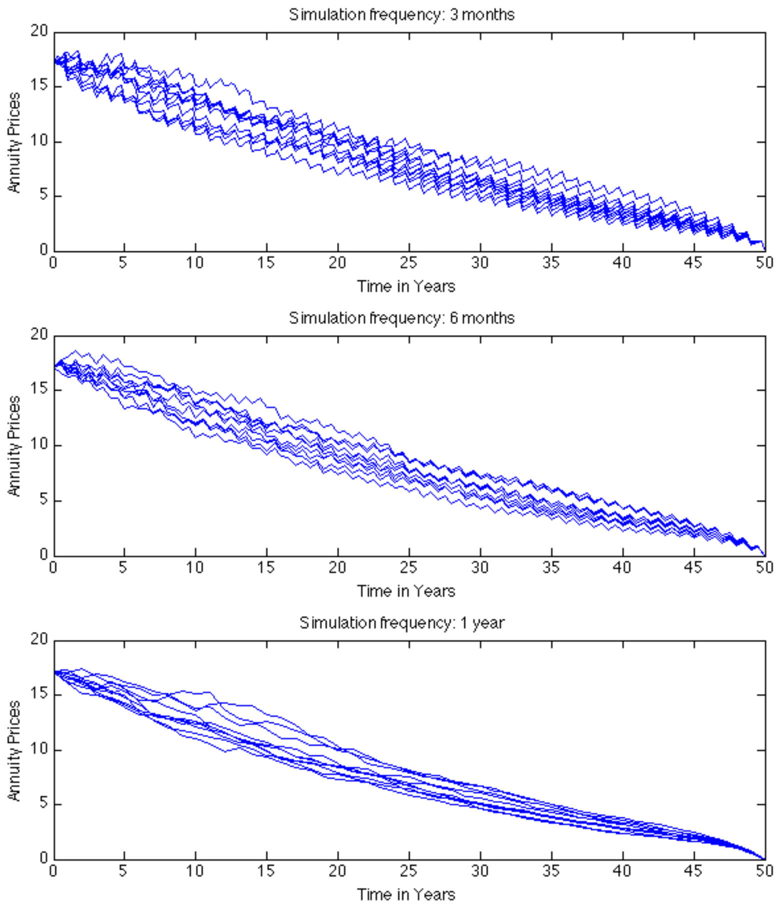

Figure 3. Simulated sample paths of a whole-life Annuity under different assumptions on the simulation step size.

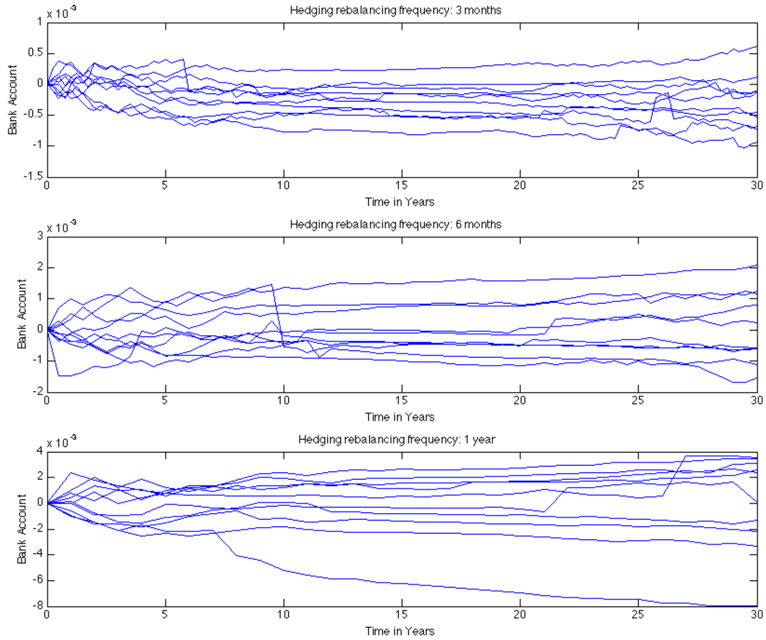

Figure 4. Simulated sample paths of the Bank Account under different assumptions on the heding rebalancing frequency.

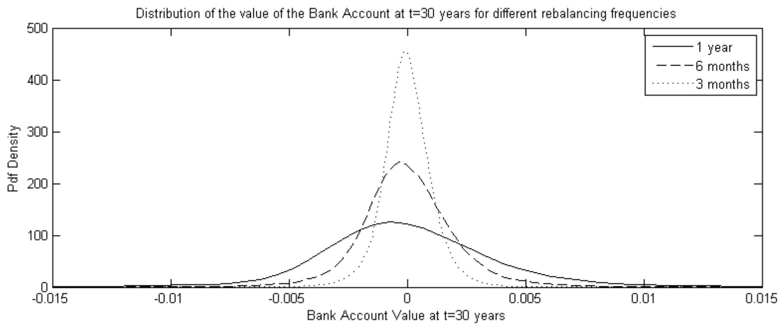

Figure 5

we neglect here and will be higher the higher the frequency. Given the results of our implementation of the dynamic hedging strategy, we compute the cost of the swap based on a value-at-risk loading principle. Table 3 reports the cost C_0 and loading m of the swap. The premium charged to the buyer of the swap C_0 is computed as the present value of the 99.5% value-at-risk of the bank account value at t=30 years obtained applying our hedging strategy with different rebalancing intervals. The resulting loading m, which represents the percentage increase in each observed survival, ranges from 0.01% to 0.04%. This value might seem low, but it is worth noticing again that it is obtained in the absence of transaction costs and basis risk, which might contribute to increase the costs of the dynamic hedging strategies.

Table 2. Hedging Error's moments under different rebalancing frequencies.

	3 months	6 months	1 year
Mean	0.0008	0.0015	0.0030
Standard Deviation	0.0011	0.0022	0.0066

Table 3. Longevity Swap premiums and loadings equivalent to the 99.5% Value-at-Risk of the Delta Gamma Hedging strategy at $t = 30$ years.

	3 months	6 months	1 year
C_0	0.0024	0.0049	0.0109
m	0.01%	0.02%	0.06%

6 Summary and further research

This paper computed the static, swap-based hedge for an annuity, and compared it with the dynamic, delta-based hedge, achieved using a longevity bond. All throughout, we assumed that the longevity intensity was distributed according to a CIR process. A similar assumption was done for the interest rate, in the theoretical part, while the empirical application focused on longevity risk. We showed that, once the model is calibrated – to a UK individual aged 65, in our case – the average hedging error of the dynamic hedge is moderate, and both its variance and the thickness of the tails of its distribution are decreasing with the rebalancing frequency, which we brought from three months to one year. We also computed the spread over the basic "swap rate" which makes 99.5% quantile of the distribution of the dynamic hedging error equal to the cost of the static hedge. This spread stayed between 0.01 and 0.04%. In doing that, we were more interested in providing a method to assess which cost of the static hedge makes it comparable to a given, tolerated error for the dynamic hedge, than to the magnitude of the result itself.

The model developed above, indeed, is novel in that it incorporates dynamic hedging in a CIR framework and its comparison with a static hedge. To fully appreciate the magnitudes of the errors one could include basis risk, i.e. the fact that static hedges are usually OTC, and therefore the reference population is the annuity one, while dynamic hedges are most likely based on indices, and therefore have a reference population which is not the annuity one. As a second refinement, one could include a number of annuitants different from one, and distinguish the idiosyncratic from the common mortality risk in the group of annuitants. Last, but most simply, we could enrich our comparison taking into consideration the fact that not only OTC-swaps are usually provided at a cost, which can be a spread on the fair rate or a fixed, initial amount, but also dynamic hedges could involve transaction costs, and the rebalancing frequency could be chosen so as to optimize – in a sense to be defined – the trade-off between the effectiveness of the hedge and its costs. We leave these three extensions to further research.

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