Decentralised Hierarchical Multi-rate Control of Large-scale Drinking Water Networks

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Abstract. We propose a decentralised hierarchical multi-rate control scheme for the control of large-scale systems with state and input constraints. The large-scale system is partitioned into sub-systems each one of which is locally controlled by a stabilizing linear controller which does not account for the prescribed constraints. A higher-level controller commands reference signals at a lower uniform sampling frequency so as to enforce linear constraints on the process variables. Worst-case subsystem interactions are modeled and accounted for in a robust manner. By optimally constraining the magnitude and rate of variation of the reference signals to each lower-level controller we prove that closed-loop stability is preserved and the fulfillment of the prescribed constraints is guaranteed. We apply the proposed methodology for the decentralised control of a quadraple-tank system, known as Johansson's system and we compare it to a centralised control approach.

1 Introduction

1.1 Motivation and Background

Large-scale systems (such as drinking water networks and power distribution networks) call for control strategies based on the spatial and temporal decomposition of the overall dynamics so as to leverage the high computational cost of a centralised control approach [1, 2]. In large scale systems hierarchical control is often the basis for a decentralised control scheme [3, 4] and various decentralised and hierarchical control schemes have been proposed in the literature for which Scattolini [5] provides a thorough review. An overview of the current architectural trends in decentralised control for large-scale interconnected systems is provided by Bakule [6].

Drinking Water Networks (DWNs) are large-scale systems whose operation is liable to set of operating, safety and quality-of-service constraints. The optimal management of DWNs is a complex task with outstanding socio-economic and environmental implications and has received considerable attention by the scientific community [7, 8]. One key reason for the use of decentralised control schemes is the need to isolate certain parts of the network for maintenance purposes without the need to re-model the overall system.

Recently, Sampathirao *et al.* [9] proposed a control framework for large-scale DWNs where pumping actions are computed by minimising a cost index. Such approaches are in the spirit of economic MPC [10], and, despite the fact that are proven to lead to improved closed-loop behaviour, may fail to guarantee the satisfaction of state constraints in closed loop. The proposed methodology allows the operator to command reference signals to the sub-systems of the network according to some cost-optimisation strategy in such a way so as to satisfy the constraints during controlled operation.

The use of reference governors has been recommended by various authors so as to mitigate the computational burden of a centralised approach by separating the constraint satisfaction problem from the stabilisation problem [11]. Recently, Kalabić and Kolmanovsky [12] proposed a methodology for the design of reference governors for constrained large-scale linear systems. Two-layer hierarchical control systems are considered in the majority of relevant publications (see [14] and references therein).

Multirate control schemes are quite popular as they increase the flexibility in the quest for the desired properties (stability, optimality, constraints satisfaction) [13–15]. A multi-rate control approach is adopted in this paper with a quantification of the effect that the ratio of the two sampling rates has on the control of the system. We will show that the adoption of different reference rates in the upper and the lower control layers offers great flexibility and enables us to strike a balance between responsiveness to set-point changes and optimality.

In this paper we propose a hierarchical multi-rate decentralised control scheme for the control of large-scale systems whose states and inputs are subject to linear constraints. The hierarchical scheme comprises two control layers: At the lower one, a linear controller stabilises the open-loop process without considering the constraints. A higher-level controller commands reference signals at a lower uniform sampling frequency so as to enforce linear constraints on the process variables. We propose a methodology for large-scale dynamically coupled linear systems which are partitioned into interconnected subsystems with state and input constraints. Worst-case interactions between subsystems are modeled and accounted for in a robust manner. By optimally constraining the magnitude and rate of variation of the reference signals to each lower-level controller, quantitative criteria are provided for selecting the ratio between the sampling rates of the upper and lower layers of control at each location, in a way that closed-loop stability is preserved and the fulfillment of the prescribed constraints is guaranteed. This paper builds on previous work by Barcelli *et al.* [16, 17] and on the ideas presented in [18].

2 Multirate Decentralised Hierarchical Control

2.1 Notation

Let $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times m}, \mathbb{N}, \mathbb{N}_{[k1,k2]}, \mathbb{S}^n_+, \mathbb{S}^n_{++}$ denote the sets of real numbers, the n-dimensional vectors, the n-by-m real matrices, the set of natural numbers, the natural numbers in the interval $[k_1, k_2]$, the set of symmetric positive semi-definite and the set of positive definite n-by-n matrices respectively. The infinity-norm of $x \in \mathbb{R}^n$ is defined as $\|x\|_\infty \triangleq \max_{i\in\mathbb{N}_{[1,n]}} |x_i|$.

Let $A \in \mathbb{R}^{n\times m}$, $\mathcal{I} \subseteq \mathbb{N}_{[1,n]}$ and $\mathcal{J} \subseteq \mathbb{N}_{[1,m]}$; we denote by $A_{\mathcal{I}\mathcal{J}} \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|}$ the submatrix of A formed by the rows and columns of A whose indices are in $\mathcal{I}$ and $\mathcal{J}$ respectively and $|\mathcal{I}|$ stands for the cardinality of the set $\mathcal{I}$. For a vector $x \in \mathbb{R}^n$, $x_{\mathcal{I}}$ denotes the vector of $\mathbb{R}^{|\mathcal{I}|}$ formed by the elements of x whose indices are in $\mathcal{I}$. We denote by $(A)_i$ the i-th row of A, while $(x)_i$ denotes the i-th element of x. Finally, we denote by 1_n the n-vector having all entries equal to 1.

2.2 Problem Formulation

The proposed setting comprises two control layers: the lower control layer (LCL) and the upper control layer (UCL) which operate at different sampling frequencies. The lower control layer comprises m independent controllers whose role is the stabilisation of the open-loop dynamics of the controlled system without taking into account the prescribed state and input constraints. The lower layer controllers operate at a higher sampling frequency, namely $1/T_L$, and receive reference signals from corresponding upper layer controllers which operate at lower sampling frequencies $1/T_H^{\langle i\rangle}$, $i{\in}\mathbb{N}_{[1,m]}$. We define $N^{\langle i\rangle} \triangleq T_H^{\langle i\rangle}/T_L$ to be the ratio between sampling frequencies of UCL and LCL which are positive integers refereed to as *reference rates*. To simplify the notation, the state variable of the system (involving all sub-systems) at the LCL sampling instants is denoted by x_k for $k \in \mathbb{N}$ (referring to all sub-systems) and the state at the UCL sampling instants is denote by $x^\nu \triangleq x_{\nu N}$ for $\nu \in \mathbb{N}$.

Let x_k, u_k, y_k respectively be the state, the input and the output of the lower layer process in discrete time and the dynamics of the system be given by:

$$x_{k+1} = \bar{A}x_k + \bar{B}u_k, \tag{1a}$$

$$y_k = \bar{C}x_k + \bar{D}u_k, \tag{1b}$$

where $x_k \in \mathbb{R}^{n_x}$, $y_k \in \mathbb{R}^{n_r}$, $u_k \in \mathbb{R}^{n_u}$ and $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are given matrices of proper dimensions.

The feedback law defining the LCL is:

$$u_k = Fx_k + Er_k, \tag{2}$$

where $r_k \in \mathbb{R}^{n_r}$ stands as a reference signal to be decided by the Upper Layer Controller (ULC).

The reference-to-output gain $\Theta \in \mathbb{R}^{n_r\times n_r}$ of (1) under feedback control law (2), is:

$$\Theta \triangleq ((\bar{C} + \bar{D}F)(I - \bar{A} - \bar{B}F)^{-1}\bar{B} + \bar{D})E. \tag{3}$$

The closed-loop system (1) can be rewritten as

$$x_{k+1} = Ax_k + Br_k, \tag{4a}$$

$$y_k = Cx_k + Dr_k, \tag{4b}$$

where $A \triangleq \bar{A} + \bar{B}F$, $B \triangleq \bar{B}E$, $C \triangleq \bar{C} + \bar{D}F$ and $D \triangleq \bar{D}E$. Additionally, matrix E must be chosen so that (A, B) is a controllable pair.

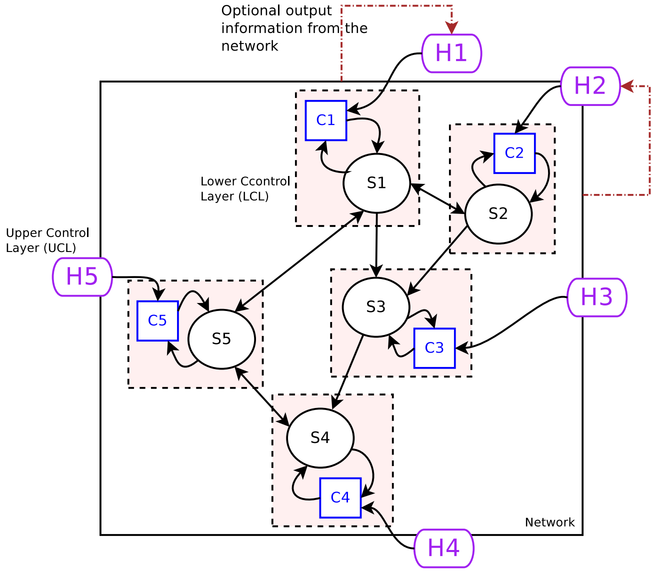

Fig. 1. Two-layer (LCL and UCL) decentralised hierarchical control scheme over a network of interconnected, dynamically coupled components. Upper-layer controllers command reference signals to the lower-layer ones which are updated at a lower frequency.

The sparsity pattern of $\bar{A}$ in (1) can be exploited so as to decompose (1) into m subsystems which are as decoupled as possible; the components of the state vector are rearranged so that $\bar{A}$ in the new coordinates is as close as possible to a block-diagonal form [19]. Let $\mathcal{I}_x^{\langle i\rangle}$, $\mathcal{I}_u^{\langle i\rangle}$ and $\mathcal{I}_r^{\langle i\rangle}$ ($i\in\mathbb{N}_{[1,m]}$) denote the sets of state, input and output indices that participate in the i-th subsystem and let $n_x^{\langle i\rangle}$, $n_u^{\langle i\rangle}$ and $n_r^{\langle i\rangle}$ be their cardinalities respectively. These sets are not assumed to be necessarily disjoint as some states and input may belong to multiple subsystems.

Assumption 1 *The pair $(\bar{A},\bar{B})$ is stabilisable and F is an asymptotically stabilizing gain for $(\bar{A},\bar{B})$ and E possess the following structure:*

$$F_{s,j}=0, \forall s\in\mathcal{I}_u^{\langle i\rangle},\ \textit{and}\ j\notin\mathcal{I}_x^{\langle i\rangle}, \forall i\in\mathbb{N}_{[1,m]}, \tag{5}$$

$$E_{s,j}=0, \forall s\in\mathcal{I}_u^{\langle i\rangle},\ \textit{and}\ j\notin\mathcal{I}_r^{\langle i\rangle}, \forall i\in\mathbb{N}_{[1,m]}. \tag{6}$$

Under Assumption 1 the LCL can be decomposed into a set of local controllers whereby the i-th controller produces the control action $u^{\langle i\rangle}\in\mathbb{R}^{n_u^{\langle i\rangle}}$ using state measurements only from the i-th subsystem according to:

$$u_k^{\langle i\rangle}=F^{\langle i\rangle}x_k^{\langle i\rangle}+E^{\langle i\rangle}r_k^{\langle i\rangle}, \tag{7}$$

where $F^{\langle i\rangle}\triangleq F_{\mathcal{I}_u^{\langle i\rangle}\mathcal{I}_x^{\langle i\rangle}}$ and $E^{\langle i\rangle}\triangleq E_{\mathcal{I}_u^{\langle i\rangle}\mathcal{I}_r^{\langle i\rangle}}$ and $x_k^{\langle i\rangle}\triangleq x_{\mathcal{I}_x^{\langle i\rangle}}$, $u_k^{\langle i\rangle}\triangleq u_{\mathcal{I}_u^{\langle i\rangle}}$ and $r_k^{\langle i\rangle}\triangleq r_{\mathcal{I}_r^{\langle i\rangle}}$ for $i\in\mathbb{N}_{[1,m]}$.

The dynamics of the different subsystems are described by the set of difference equations:

$$\Sigma^{\langle i\rangle} : x_{k+1}^{\langle i\rangle} = A^{\langle i\rangle} x_k^{\langle i\rangle} + B^{\langle i\rangle} r_k^{\langle i\rangle} + d_k^{\langle i\rangle}, \tag{8}$$

where $A^{\langle i\rangle} \triangleq A_{\mathcal{I}_x^{\langle i\rangle}\mathcal{I}_x^{\langle i\rangle}}$, $B^{\langle i\rangle} \triangleq B_{\mathcal{I}_x^{\langle i\rangle}\mathcal{I}_r^{\langle i\rangle}}$ and $d_k^{\langle i\rangle}$ is a disturbance term to compensate for the unmodeled dynamics due to neglected state couplings between the subsystem $\Sigma^{\langle i\rangle}$ and its neighbours. The gains $F^{\langle i\rangle}$ are chosen so that the subsystems $\Sigma^{\langle i\rangle}$ are open-loop stable (with $r_k \equiv 0$ and $d_k \equiv 0$).

Assumption 2 *In addition to Assumption 1, for every $i \in \mathbb{N}_{[1,m]}$ the feedback gain $F^{\langle i\rangle}$ stabilises subsystem $\Sigma^{\langle i\rangle}$.*

Various methodologies have been proposed for the computation of such sparse stabilizing gains [20, 21].

Let us define $\mathcal{J}_x^{\langle i\rangle} \triangleq \mathbb{N}_{[1,n_x]} \setminus \mathcal{I}_x^{\langle i\rangle}$, and $\mathcal{J}_r^{\langle i\rangle} \triangleq \mathbb{N}_{[1,n_r]} \setminus \mathcal{I}_r^{\langle i\rangle}$. The vectors $\tilde{x}^{\langle i\rangle} \triangleq x_{\mathcal{J}_x^{\langle i\rangle}}$ and $\tilde{r}^{\langle i\rangle} \triangleq r_{\mathcal{J}_r^{\langle i\rangle}}$ will be referred to as *neglected* states and references. The pair $(\tilde{A}^{\langle i\rangle}, \tilde{B}^{\langle i\rangle})$ with $\tilde{A}^{\langle i\rangle} \triangleq A_{\mathcal{I}_x^{\langle i\rangle}\mathcal{J}_x^{\langle i\rangle}}$ and $\tilde{B}^{\langle i\rangle} \triangleq B_{\mathcal{I}_x^{\langle i\rangle}\mathcal{J}_r^{\langle i\rangle}}$ will be used to describe the effect of the neglected states and references on the system $\Sigma^{\langle i\rangle}$.

Then the UCL comprises m subcontrollers which produce the reference signals $r_k^{\langle i\rangle}$ so as to keep the state $x^{\langle i\rangle}$ and the reference $r^{\langle i\rangle}$ inside the polytope:

$$\mathcal{Z}^{\langle i\rangle} \triangleq \{ \left[\begin{smallmatrix} x \\ r \end{smallmatrix}\right] \in \mathbb{R}^{n_x^{\langle i\rangle}+n_r^{\langle i\rangle}} : H_x^{\langle i\rangle} x + H_r^{\langle i\rangle} r \leq K^{\langle i\rangle} \}, \tag{9}$$

where $H_x^{\langle i\rangle} \in \mathbb{R}^{q_i \times n_x^{\langle i\rangle}}$, $H_r^{\langle i\rangle} \in \mathbb{R}^{q_i \times n_r^{\langle i\rangle}}$, and $K^{\langle i\rangle} \in \mathbb{R}^{q_i}$. The overall set of constraints is then defined as $\mathcal{Z} \triangleq \{ \left[\begin{smallmatrix} x \\ r \end{smallmatrix}\right] \in \mathbb{R}^{n_x+n_r} : (x^{\langle i\rangle}, r^{\langle i\rangle}) \in \mathcal{Z}^{\langle i\rangle}, \forall i \in \mathbb{N}_{[1,m]} \}$.

Let $A_0^{\langle i\rangle} \in \mathbb{R}^{n_x^{\langle i\rangle} \times n_x}$ be the matrix obtained by collecting the rows of A with indices in $\mathcal{I}_x^{\langle i\rangle}$ and setting to zero the elements in the columns $\mathcal{I}_x^{\langle i\rangle}$. Similarly, we construct $B_0^{\langle i\rangle} \in \mathbb{R}^{n_x^{\langle i\rangle} \times n_r}$ by collecting from B the rows indexed by $\mathcal{I}_x^{\langle i\rangle}$ and then zeroing the columns whose index is in $\mathcal{I}_r^{\langle i\rangle}$. Then, it holds that:

$$x_{k+1}^{\langle i\rangle} = A^{\langle i\rangle} x_k^{\langle i\rangle} + B^{\langle i\rangle} r_k^{\langle i\rangle} + A_0^{\langle i\rangle} x_k + B_0^{\langle i\rangle} r_k. \tag{10}$$

Additionally, let us define the set $\mathcal{Z} \triangleq \{(x,r) : (x^{\langle i\rangle}, r^{\langle i\rangle}) \in \mathcal{Z}^{\langle i\rangle}, \forall i \in \mathbb{N}_{[1,m]}\}$, which is a polytope and can be written in the form $\mathcal{Z} = \{(x,r) : H_x x + H_r r \leq K\}$. Let the reference vector $r^{\langle i\rangle}$ be constrained in the set:

$$\mathcal{R}^{\langle i\rangle} \triangleq \{ r^{\langle i\rangle} \in \mathbb{R}^{n_r^{\langle i\rangle}} : (H_x^{\langle i\rangle} G^{\langle i\rangle} + H_r^{\langle i\rangle}) r^{\langle i\rangle} \leq K^{\langle i\rangle} - \Delta K^{\langle i\rangle} \},$$

where $G^{\langle i\rangle} \triangleq (I - A^{\langle i\rangle})^{-1} B^{\langle i\rangle}$ is the reference-to-state static gain for $\Sigma^{\langle i\rangle}$ and $\Delta K^{\langle i\rangle} \geq 0$. We assume that the reference signals $r_k^{\langle i\rangle}$ retain the tracking error $\Delta x_k^{\langle i\rangle} \triangleq x_k^{\langle i\rangle} - G^{\langle i\rangle} r_k^{\langle i\rangle}$ in the set:

$$\mathcal{E}^{\langle i\rangle} = \{ \Delta x^{\langle i\rangle} \in \mathbb{R}^{n_x^{\langle i\rangle}} : H_x^{\langle i\rangle} \Delta x^{\langle i\rangle} \leq \Delta K^{\langle i\rangle} \}. \tag{11}$$

Notice that $\Delta x_k^{\langle i\rangle} \in \mathcal{E}^{\langle i\rangle}$ if and only if $(x_k^{\langle i\rangle}, r_k^{\langle i\rangle}) \in \tilde{\mathcal{E}}^{\langle i\rangle}$ where:

$$\tilde{\mathcal{E}}^{\langle i\rangle} \triangleq \{\left[\begin{smallmatrix} x^{\langle i\rangle} \\ r^{\langle i\rangle}\end{smallmatrix}\right] \in \mathbb{R}^{n_x^{\langle i\rangle}+n_r^{\langle i\rangle}} : x^{\langle i\rangle} - G^{\langle i\rangle} r^{\langle i\rangle} \in \mathcal{E}^{\langle i\rangle}\}. \tag{12}$$

If we set $z^{\langle i\rangle} \triangleq G^{\langle i\rangle} r^{\langle i\rangle} = A^{\langle i\rangle} z^{\langle i\rangle} + B^{\langle i\rangle} r^{\langle i\rangle}$, then the dynamics of $\Sigma^{\langle i\rangle}$ can be described in terms of $\Delta x^{\langle i\rangle} = x^{\langle i\rangle} - z^{\langle i\rangle}$ as follows:

$$\Delta x_{k+1}^{\langle i\rangle} = A^{\langle i\rangle} \Delta x_k^{\langle i\rangle} + d_k^{\langle i\rangle}, \tag{13}$$

where, under the assumptions that $(x_k^{\langle i\rangle}, r_k^{\langle i\rangle}) \in \mathcal{Z}^{\langle i\rangle}$ and $\Delta x_k^{\langle i\rangle} \in \mathcal{E}^{\langle i\rangle}$ for all $k \in \mathbb{N}$ and $i \in \mathbb{N}_{[1,m]}$, the disturbance $d_k^{\langle i\rangle}$ is drawn from the polytope:

$$\mathcal{D}^{\langle i\rangle} = \left\{ d^{\langle i\rangle} \in \mathbb{R}^{n_x^{\langle i\rangle}} \middle| \begin{array}{l} \exists r \in \mathbb{R}^{n_r}, \exists x \in \mathbb{R}^{n_x}, \text{s.t.:} d^{\langle i\rangle} = A_0^{\langle i\rangle} x + B_0^{\langle i\rangle} r, \\ \text{and } \forall j \in \mathbb{N}_{[1,m]} : (x^{\langle j\rangle}, r^{\langle j\rangle}) \in \mathcal{Z}^{\langle j\rangle} \cap \tilde{\mathcal{E}}^{\langle j\rangle} \end{array} \right\}. \tag{14}$$

The size of this polytope determines how strongly the i-th subsystem is dynamically coupled with its neighbours.

Let $\Omega^{\langle i\rangle}(0)$ be the maximal robustly positive invariant set for (13) under the constraints $\Delta x^{\langle i\rangle} \in \mathcal{E}^{\langle i\rangle}$ and for $d_k^{\langle i\rangle} \in \mathcal{D}^{\langle i\rangle}$ for all $k \in \mathbb{N}$. Let $\Omega^{\langle i\rangle}(0)$ have the minimal representation $\Omega^{\langle i\rangle}(0) = \{x \in \mathbb{R}^{n_x^{\langle i\rangle}} : H_0^{\langle i\rangle} x \leq K_0^{\langle i\rangle}\}$, counting $n_0^{\langle i\rangle}$ inequalities. Under Assumption 2 this set exists and is a finitely generated polytope.

The complexity of the computation of a maximal RPI set for the overall large-scale system can prove preventive even for offline computations. Note, however, that the computation of the maximal RPI sets is done in a decentralised fashion. For $r \in \mathcal{R}^{\langle i\rangle}$ we define the sets $\Omega^{\langle i\rangle}(r) \triangleq \{x \in \mathbb{R}^{n_x^{\langle i\rangle}} : x - G^{\langle i\rangle} r \in \Omega^{\langle i\rangle}(0)\}$.

The following theorem is the main result of this section and provides an invariance result for hierarchical multi-rate control systems.

Theorem 1 *For all* $i \in \mathbb{N}_{[1,m]}$ *let* $x_0^{\langle i\rangle} \in \Omega^{\langle i\rangle}(r^{\langle i\rangle})$ *and assume that* $r_k^{\langle i\rangle} = r^{\langle i\rangle} \in \mathcal{R}^{\langle i\rangle}$ *for all* $k \in \mathbb{N}$. *Then* $(x_k^{\langle i\rangle}, r_k^{\langle i\rangle}) \in \mathcal{Z}^{\langle i\rangle}$ *for all* $k \in \mathbb{N}$ *and* $i \in \mathbb{N}_{[1,m]}$.

2.3 Computation of Maximum Reference Variations

Assume that a set of fixed reference rates $N^{\langle i\rangle}$ for $i \in \mathbb{N}_{[1,m]}$ is given. In this section we will compute upper bounds on the element-wise variations of the reference rates $r^{\langle i\rangle}$ so that $(x_k^{\langle i\rangle}, r_k^{\langle i\rangle})$ satisfies the prescribed constraints (9). For every subsystem $i \in \mathbb{N}_{[1,m]}$ we formulate the problem of determining the minimum element-wise change in the reference signal that may lead the initial state $x_{\nu N}^{\langle i\rangle}$ outside $\Omega^{\langle i\rangle}(r^{\langle i\rangle,\nu})$; the problem is stated as follows:

$$\mathbb{P}_N^{\langle i\rangle} \colon \rho^{\langle i\rangle}(N) \triangleq \min_{r^1, r^2, x_0, d_0, \ldots, d_{N-1}} \|r^1 - r^2\|_\infty, \tag{15a}$$

subject to:

$$r^1, r^2 \in \mathcal{R}^{\langle i \rangle}, \tag{15b}$$

$$x_0 \in \Omega^{\langle i \rangle}(r^1), \tag{15c}$$

$$d_j^{\langle i \rangle} \in \mathcal{D}^{\langle i \rangle}, \forall j \in \mathbb{N}_{[0,N-1]}, \tag{15d}$$

$$(A^{\langle i \rangle})^N x_0 + \Gamma_N^{\langle i \rangle} r^2 + \sum_{j=0}^{N-1} (A^{\langle i \rangle})^{N-j-1} d_j^{\langle i \rangle} \notin \Omega^{\langle i \rangle}(r^2), \tag{15e}$$

where $\Gamma_N^{\langle i \rangle} \triangleq \sum_{j=0}^{N-1} (A^{\langle i \rangle})^j B^{\langle i \rangle}$. The above optimisation problem can be formulated as a MILP.

The value function of (15a) enjoys a very useful property: it is non-decreasing with respect to N. If $\mathbb{P}_N$ is infeasible for some N, this implies that for all $r^{\nu-1}, r^\nu \in \mathcal{R}$ it is $x^{\nu+1} \in \Omega(r^\nu)$ whenever $x^\nu \in \Omega(r^{\nu-1})$. In this case we set $\rho(N) = \infty$.

The following theorem states the conditions under which the constraints are satisfied in closed-loop. Note that, except for the last consequence of the theorem, no convergence of the system's trajectories to some constant value is assumed or required. This suggests that a purely cost-driven approach can be applied where the system's trajectories move in an oscillatory manner leading to an economically profitable performance determined by the optimisation of a performance criterion in a receding horizon fashion [9, 10].

Theorem 2 *Let F be a (decentralised) asymptotically stabilizing gain satisfying Assumption 2. Assume that for every subsystem $i \in \mathbb{N}_{[1,m]}$ there is a $\sigma^{\langle i \rangle} > 0$ so that the references $r^{\langle i \rangle,\nu}$ produced by the upper-layer controllers satisfy the following rate constraint at all time instants $\nu \in \mathbb{N}$:*

$$\|r^{\langle i \rangle,\nu} - r^{\langle i \rangle,\nu-1}\|_\infty \leq \rho^{\langle i \rangle}(N^{\langle i \rangle}) - \sigma^{\langle i \rangle}, \tag{16a}$$

$$r^{\langle i \rangle,\nu-1}, r^{\langle i \rangle,\nu} \in \mathcal{R}^{\langle i \rangle}. \tag{16b}$$

Let $x_0^{\langle i \rangle} \in \Omega^{\langle i \rangle}(r^{-1,\langle i \rangle})$ for all $i \in \mathbb{N}_{[1,m]}$. Then the linear system (1) with the feedback control law (2) satisfies the the constraints $\left[\begin{smallmatrix} x_k \\ r_k \end{smallmatrix}\right] \in \mathcal{Z}$ for all $k \in \mathbb{N}$. Additionally, if $\lim_{k\to\infty} r_k = r$ with $r \in \mathcal{R}$, then $\lim_{k\to\infty} x_k = Gr$.

The UCL control action can be computed by a model predictive control strategy where any optimality criterion can be used so long as the constraints (16) are satisfied.

3 Control of a System of Interconnected Tanks

3.1 System dynamics and decomposition

The proposed methodology is tested on Johansson's quadruple-tank process [22] where the control objective is to track given (possibly time-varying, piece-wise constant) references s_1 and s_2 for the levels of tanks 1 and 2, namely h_1 and h_2, as in Fig. 2 by manipulating the inflows q_a and q_b. Constraints are imposed on the maximum flow that can be achieved by each pump and on the upper and lower allowed levels of water in the tanks.

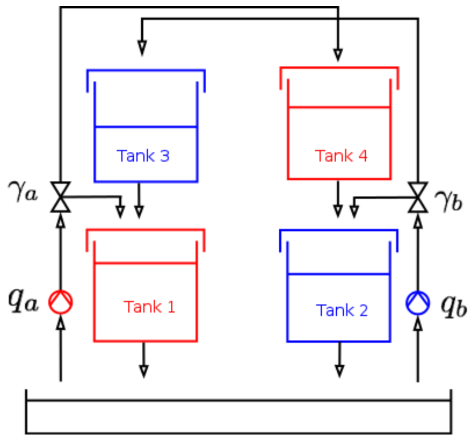

Fig. 2. Johansson's quadruple-tank process where the two sub-systems are denoted with different colours.

The system is subject to state and input constraints and its dynamics is described in [23] by the system of continuous-time nonlinear equations

$$S_1 \frac{\mathrm{d}h_1}{\mathrm{d}t} = -a_1\sqrt{2gh_1} + a_3\sqrt{2gh_3} + \gamma_a q_a, \tag{17a}$$

$$S_2 \frac{\mathrm{d}h_2}{\mathrm{d}t} = -a_2\sqrt{2gh_2} + a_4\sqrt{2gh_4} + \gamma_b q_b, \tag{17b}$$

$$S_3 \frac{\mathrm{d}h_3}{\mathrm{d}t} = -a_3\sqrt{2gh_3} + (1-\gamma_b) q_b, \tag{17c}$$

$$S_3 \frac{\mathrm{d}h_4}{\mathrm{d}t} = -a_4\sqrt{2gh_4} + (1-\gamma_a) q_a. \tag{17d}$$

The maximum allowed level for tanks 1 and 2 is set to 1.36 m and for tanks 3 and 4 to 1.30 m. The minimum allowed level in all tanks is 0.2 m. The maximum flows are $q_{a,\max} = 3.26\ \mathrm{m}^3/\mathrm{h}$ and $q_{b,\max} = 4\ \mathrm{m}^3/\mathrm{h}$; no negative flows are possible. The values of the other parameters of the system are $a_1 = 1.31 \cdot 10^{-4}\ \mathrm{m}^2$, $a_2 = 1.51 \cdot 10^{-4}\ \mathrm{m}^2$, $a_3 = 9.27 \cdot 10^{-5}\ \mathrm{m}^2$, $a_4 = 8.82 \cdot 10^{-5}\ \mathrm{m}^2$, $S_1 = S_2 = 0.06\ \mathrm{m}^2$, $S_3 = S_4 = 0.20\ \mathrm{m}^2$, and $\gamma_a = \gamma_b = 0.5$. The nonlinear system is linearised about the steady state $u^0 = (2.6, 2.6)'\ \mathrm{m}^3/\mathrm{h}$ and $x^0 = (0.6545, 0.4926, 0.7852, 0.8583)'$ m and discretised with sampling period $T_s = 10$s. We define the discrete-time state vector $x_k = (h_{1,k}, h_{2,k}, h_{3,k}, h_{4,k})'$ which comprises the levels of the four tanks, the discrete-time input vector $u_k = (q_{a,k}, q_{b,k})'$ of manipulated variables which are the two flows, and the discrete output $y_k = x_k$. The linearised discrete-time system is written in the form of (1).

3.2 Centralised versus decentralised control

We consider that the lower control layer operates at sampling time $T_s = 10$s. The overall system is partitioned into two subsystems with $\mathcal{I}_x^{\langle 1\rangle} = \{1,4\}$, $\mathcal{I}_x^{\langle 2\rangle} = \{2,3\}$ and $\mathcal{I}_u^{\langle 1\rangle} = \{1\}$, $\mathcal{I}_u^{\langle 2\rangle} = \{2\}$. The system is controlled by means of the proposed decentralised hierarchical control methodology which is compared to its centralised hierarchical variant. Reference commands from the upper layer controller are computed so that they minimise a quadratic cost function. In particular, the UCL for subsystem 1 solves the following minimisation problem at the UCL sampling time instant ν:

$$J^{\langle 1\rangle\star}(x^\nu) = \min_{\{r_1^{\nu+j}\}_{j=0}^{N-1}} \sum_{k=0}^{N-1} (h_1^{\nu+k} - s_1)^2 + \lambda(r_a^{\nu+k} - r_a^s)^2, \tag{18}$$

subject to the (linearised) system dynamics, measurements from the system, the requirement $r^{\nu+k} \in \mathcal{R}$ for all $k = 0, \dots, N-1$, and the bounds on the maximum reference variation that accrue from Theorem 2. In what follows, the weight λ is fixed to 0.01. Then, the solution of problem 18 yields an optimal sequence of references $\{r_1^{\nu+k,\star}\}_{k=0}^{N-1}$, the first one of which – namely $r_1^{\nu,\star}$ is applied to the corresponding controlled LCL system in a receding horizon fashion. The UCL controller for sub-system 2 works in an analogous fashion where the minimisation problem becomes $J^{\langle 2\rangle\star}(x^\nu) = \min_{\{r_2^{\nu+j}\}_{j=0}^{N-1}} \sum_{k=0}^{N-1} (h_2^{\nu+k} - s_2)^2 + \lambda(r_b^{\nu+k} - r_b^s)^2$, subject to the corresponding constraints. According to Theorem 2 the closed-loop system will satisfy the prescribed constraints.

For the decentralised control case, the dependence of the maximum reference change $\rho^{\langle i\rangle}$ on N is presented in Figure 3. The reference rate $N = 40$ was selected for which $\rho^{\langle 1\rangle}(N) = 0.0034$ and $\rho^{\langle 2\rangle}(N) = 0.0063$ for the decentralised control system and $\rho(N) = 0.0035$ for the centralised control approach. The maximum reference variation $\rho^{\langle i\rangle}(N)$ for the two subsystems is presented in Figure 3. Notice that for $N \geq N^\star = 42$, it is $\rho^{\langle 2\rangle}(N) = \infty$. Vectors $\Delta K^{\langle i\rangle}$ in (11) were chosen to be $\Delta K^{\langle i\rangle} = cK^{\langle i\rangle}$, with $c = 0.5$.

The controlled trajectories of the tank levels are presented in Figures 4 to 5. The tank levels h_1 and h_2 are steered towards four different set-points and the set-point values are kept constant for 5.55h. In order to quantify the performance of the three controllers, we use the following index introduced by Alvarado *et al.* [23] for the same system:

$$J = \sum_{k=0}^{N_s-1} (h_{1,k} - s_{1,k})^2 + (h_{2,k} - s_{2,k})^2 + \kappa((q_{a,k} - q_{a,k}^s)^2 + (q_{b,k} - q_{b,k}^s)^2), \tag{19}$$

where $\kappa = 0.01$ and $q_{a,k}^s$ and $q_{b,k}^s$ are the steady-state values of the input variables that correspond to the set-point defined by s_1 and s_2, and $N_s = 8000$ $(22h)$ is

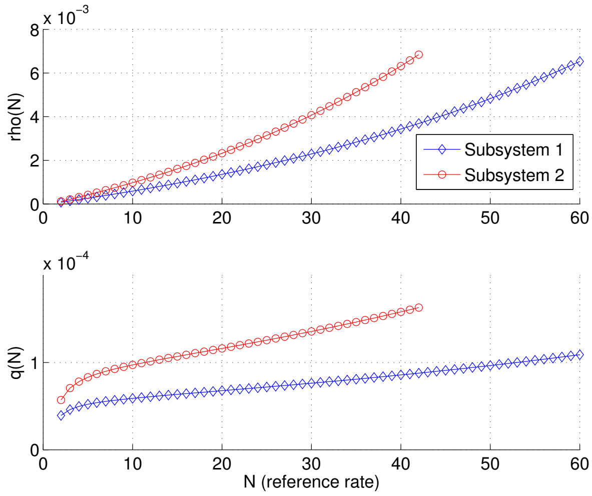

Fig. 3. The functions $\rho^{\langle i \rangle}(N)$ and $q^{\langle i \rangle}(N) \triangleq \rho^{\langle i \rangle}(N)/N$ for the two subsystems.

Controller	$\tau_{s,1}$ (h)	$\tau_{s,2}$ (h)	J
DHMPC	0.1674	0.1500	0.1495
CHMPC	0.1146	0.1458	0.1516

Table 1. Performance of a decentralised and a centralised controller for Johansson's system.

the simulation horizon. The values of the performance index J are presented in Table 1.

The maximal robust positive invariant sets $\Omega^{\langle i \rangle}(0)$, $i \in \{1, 2\}$ for the decentralised control case were computed offline in 1.97s and 2.19s and their minimal representations involved 5 and 4 inequalities respectively. The maximal positive invariant set $\Omega(0)$ for the centralised control system was computed in 0.60s and its minimal representation comprised 12 linear inequalities. The associated MILPs $\mathbb{P}_N^{\langle i \rangle}$ as in (15) were solved offline in $2.12s$ for subsystem 1 and $2.27s$ for subsystem 2 on average. The corresponding centralised computation required $6.33s$ on average. All reported computation times were measured in MATLAB 2013a running on a Mac OS X machine, 2.66GHz Intel Core 2 Duo, 4GB RAM.

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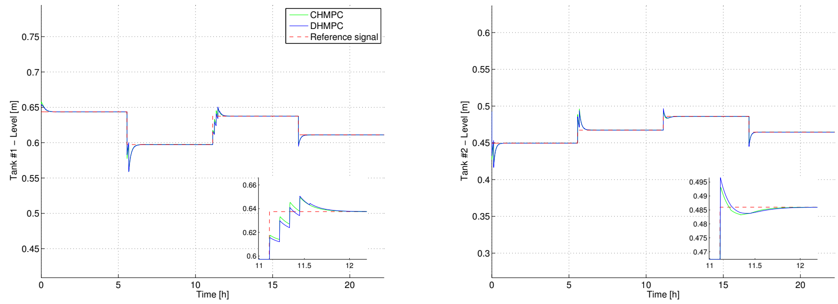

Fig. 4. The level in tank 1 and 2: Comparison between centralised hierarchical MPC (CHMPC, green) and decentralised hierarchical MPC (DHMPC, blue). The dashed red line represents the set-point s_1. The inset shows the convergence of the tank level to the desired set-point in the interval 11 to 12.2h.

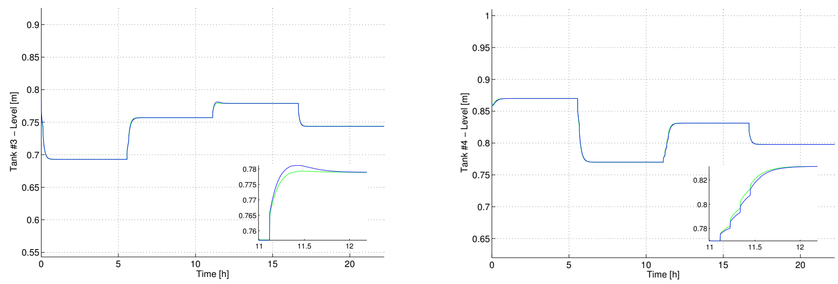

Fig. 5. The level in tanks 3 and 4: Closed-loop trajectories for CHMPC (green), and DHMPC (blue).

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