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AGING FUNCTIONS AND MULTIVARIATE NOTIONS OF NBU AND IFR

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For $d \geq 2$, let $\mathbf{X} = (X_1, \ldots, X_d)$ be a vector of exchangeable continuous lifetimes with joint survival function $\overline{F}$. For such models, we study some properties of multivariate aging of $\overline{F}$ that are described by means of the multivariate aging function $B_{\overline{F}}$, which is a useful tool for describing the level curves of $\overline{F}$. Specifically, the attention is devoted to notions that generalize the univariate concepts of New Better than Used and Increasing Failure Rate. These multivariate notions are satisfied by random vectors whose components are conditionally independent and identically distributed having univariate conditional survival function that is New Better than Used (respectively, Increasing Failure Rate). Furthermore, they also have an interpretation in terms of comparisons among conditional survival functions of residual lifetimes, given a same history of observed survivals.

1. INTRODUCTION

For $d \in \mathbb{N}, d \geq 2$, let $\mathbf{X} = (X_1, \ldots, X_d)$ be a vector of continuous and positive random variables with joint survival function $\overline{F} : \overline{\mathbb{R}}_+^d \to \mathbb{I}$, where $\overline{\mathbb{R}}_+ = [0, +\infty]$ and $\mathbb{I} = [0, 1]$. In the field of reliability, where $X_1, \ldots, X_d$ are interpreted as *lifetimes* of units

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or of components in a system, one is generally interested in studying qualitative properties of $\overline{F}$, such as dependence properties and aging properties.

As is well known, several approaches have been proposed in the literature to define properties of multivariate aging that could be considered as natural extensions of the univariate aging notions.

Here we focus our attention on the case when $\mathbf{X}$ is exchangeable and, consequently, the univariate survival margins are equal to a same univariate survival function that we denote by $\overline{G}$, $\overline{G} : \overline{\mathbb{R}}_+ \to \mathbb{I}$.

For such models, we aim at studying some properties of multivariate aging of $\overline{F}$. In particular, we will consider properties that are described by means of the *multivariate aging function* $B_{\overline{F}} : \mathbb{I}^d \to \mathbb{I}$, given by

$$B_{\overline{F}}(\mathbf{u}) = \exp\Big(-\overline{G}^{-1}(\overline{F}(-\log(u_1), \ldots, -\log(u_d)))\Big). \tag{1.1}$$

Such a $B_{\overline{F}}$ is a useful tool for describing the level curves of $\overline{F}$ and, as discussed in some previous articles (see [6,7,9,16,18]), it can be used in fact for investigating some notions of multivariate aging. Specifically, in [6] (see also [9]), it has been argued that notions of multivariate aging based on $B_{\overline{F}}$ can be defined by means of the following scheme:

(i) Consider a univariate aging notion P (e.g., New Better than Used (NBU), Increasing Failure Rate (IFR)).

(ii) Take the joint survival function $\overline{F}$ of d independent and identically distributed (i.i.d.) lifetimes and prove results of the following type: Each lifetime has the property P if, and only if, $B_{\overline{F}}$ has the property $\widetilde{P}$.

(iii) Define a multivariate aging notion as follows: Any exchangeable survival function $\overline{F}$ is multivariate-P if $B_{\overline{F}}$ has the property $\widetilde{P}$.

In [6,9], the above analysis has been developed for the case $d = 2$, where it is also shown that, for notions of this type, the relations among univariate aging, multivariate aging, and dependence properties of $\overline{F}$ can be easily analyzed. In this article, we aim at pointing out features and differences that arise in the extension of this study to the multivariate case, making it worthy of further analysis.

Specifically, we concentrate our attention on notions that generalize the univariate concepts of NBU and IFR. As we will show, the multivariate notions to be introduced are satisfied by random vectors whose components are conditionally i.i.d. having a NBU (respectively, IFR) univariate conditional survival function. This circumstance has been considered as a natural requirement for Bayesian notions of multivariate aging (see, e.g., [2]). Moreover, it implies the usual assumption that multivariate extensions of some univariate aging property P should be satisfied by vectors of i.i.d. lifetimes of type P.

Furthermore, these notions also have an interpretation in terms of the comparisons among conditional survival functions of residual lifetimes, given the same history of observed survivals, another interesting property of Bayesian aging according to [4].

The article is organized as follows. Section 2 contains basic definitions and properties of multivariate aging functions. Some subclasses of such functions are also introduced and the relations among them discussed. Section 3 presents some definitions of multivariate aging extending notions of NBU and IFR. Their properties are discussed in detail. Finally, Section 4 is devoted to a short discussion about the introduced definitions and their related results.

2. MULTIVARIATE AGING FUNCTION: DEFINITIONS AND PROPERTIES

We start this section by introducing some useful notations and definitions.

Through this article, we will often formulate our results referring to one of the following assumptions:

ASSUMPTION 1 (Exchangeable Case): *We consider an exchangeable random vector* $\mathbf{X} = (X_1, \ldots, X_d)$ $(d \geq 2)$ *of continuous lifetimes with joint survival function* $\overline{F} : \overline{\mathbb{R}}_+^d \to \mathbb{I}$ *and univariate survival marginals equal to* $\overline{G}$. *We suppose that* $\overline{G}$ *is strictly decreasing on* $\overline{\mathbb{R}}_+$ *with* $\overline{G}(0) = 1$ *and* $\overline{G}(+\infty) = 0$.

ASSUMPTION 2 (i.i.d. Case): *Under Assumption 1, we suppose in addition that* $X_1, X_2, \ldots, X_d$ *are independent.*

Given every survival function $\overline{F}$, we can uniquely define the *survival copula* $K_{\overline{F}} : \mathbb{I}^d \to \mathbb{I}$ by means of the formula

$$K_{\overline{F}}(u_1, \ldots, u_d) = \overline{F}\left(\overline{G}^{-1}(u_1), \ldots, \overline{G}^{-1}(u_d)\right).$$

It can also be defined, in an implicit way, by means of the functional equation

$$\overline{F}(x_1, \ldots, x_d) = K_{\overline{F}}(\overline{G}(x_1), \ldots, \overline{G}(x_d)).$$

Such a $K_{\overline{F}}$ describes the dependence properties of $\mathbf{X}$ (see, e.g., [19,25]). When the components of $\mathbf{X}$ are independent, $K_{\overline{F}}(\mathbf{u}) = \Pi_d(\mathbf{u}) = u_1 u_2 \cdots u_d$; in this case, we will also denote $\overline{F}$ by means of the symbol $\overline{F}_{\Pi,\overline{G}}$.

Associated with $\overline{F}$, we can also consider the *multivariate aging function* $B_{\overline{F}}$ as given by (1.1). In terms of the copula $K_{\overline{F}}$, we can write

$$B_{\overline{F}}(\mathbf{u}) = \exp\Big(-\overline{G}^{-1}(K_{\overline{F}}(\overline{G}(-\log(u_1)), \ldots, \overline{G}(-\log(u_d)))\Big). \quad \textbf{(2.1)}$$

We will denote by $B_{\Pi,\overline{G}}$ the multivariate aging function corresponding to the copula Π_d, namely

$$B_{\Pi,\overline{G}}(\mathbf{u}) = \exp\Big(-\overline{G}^{-1}(\overline{G}(-\log(u_1)) \cdots \overline{G}(-\log(u_d)))\Big). \quad \textbf{(2.2)}$$

The function $B_{\overline{F}}$ is an object allowing us to define certain aging properties of $\overline{F}$ (see [6,9,16]). It is obtained by transforming the copula $K_{\overline{F}}$ by means of the bijection

$\overline{G} \circ (-\log)$ of $\mathbb{I}$. Transformations of this type are known as *distortions* of copulas (see [10,14,21] and the references therein).

Notice that whereas a copula K is a probability distribution function (concentrating the probability mass on $\mathbb{I}^d$), a multivariate aging function B might not be of this type. In fact, B satisfies the following properties:

(B1) $B(\mathbf{u}) = u_i$ for any $\mathbf{u} \in \mathbb{I}^d$ having all the components equal to 1 except possibly for the ith one.

(B2) B is increasing in each variable.

However, B need not be d-increasing; in other words, B is a *semicopula* but not necessarily a copula (see, e.g., [9,15,16]).

In the sequel, we use the term "multivariate aging function" to denote any continuous function $B : \mathbb{I}^d \to \mathbb{I}$ satisfying (B1) and (B2) that can be obtained from some survival function $\overline{F}$ by means of (1.1). Note that every copula K is a multivariate aging function, since it can be obtained as the multivariate aging function of a survival function $\overline{F}$ having copula K and univariate survival marginal $\overline{G}(t) = \exp(-t)$.

Within the family of the multivariate aging functions, we define the following classes; as we will see, these classes will be used to express our multivariate aging notions.

DEFINITION 2.1: *Let B be a multivariate aging function. We say that:*

(A1) $B \in \mathscr{A}_1^+$ *if, and only if, for every* $\mathbf{u} \in \mathbb{I}^d$,

$$B(u_1, \ldots, u_d) \geq \Pi_d(u_1, \ldots, u_d). \tag{2.3}$$

(A2) $B \in \mathscr{A}_2^+$ *if, and only if, for all* $i, j \in \{1, \ldots, d\}$, $i \neq j$, *and for every* $\mathbf{u} \in \mathbb{I}^d$,

$$B(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_d) \geq B(u_1, \ldots, u_i u_j, \ldots, 1, \ldots, u_d). \tag{2.4}$$

(A3) $B \in \mathscr{A}_3^+$ *if, and only if, for all* $i, j \in \{1, \ldots, d\}$, $i \neq j$, *for all* $u_i, u_j \in \mathbb{I}$, $u_i \geq u_j$, *and for every* $s \in (0, 1)$,

$$B(u_1, \ldots, u_i s, \ldots, u_j, \ldots, u_d) \geq B(u_1, \ldots, u_i, \ldots, u_j s, \ldots, u_d). \tag{2.5}$$

The corresponding classes $\mathscr{A}_i^-$ $(i = 1, 2, 3)$ *are defined by reversing the inequality signs in* (2.3), (2.4), *and* (2.5), *respectively.*

The property of (2.3) is a pointwise comparison between the multivariate aging function B and the copula Π_d. In particular, copulas satisfying (2.3) are called *positive lower orthant dependent* (see [25]). Properties expressed in (2.4) and (2.5) are essentially inequalities related to the bivariate sections of B. In particular, (2.5) consists of the *supermigrativity* (compare with [11]) of all the bivariate sections of B, whereas (2.4) is one of its weaker forms, obtained by letting $u_i = 1$ and $s = 1/u_j$ in (2.5). Therefore, $\mathscr{A}_3^+ \subseteq \mathscr{A}_2^+$; however, the converse inclusion is not true, as will be shown in Example 2.4.

Furthermore, $\mathscr{A}_2^+ \subseteq \mathscr{A}_1^+$. In fact, by iteratively applying (2.4), we obtain that, for every $\mathbf{u} \in \mathbb{I}^d$,

$$
\begin{aligned}
B&(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_k, \ldots, u_d)\\
&\geq B(u_1, \ldots, u_i u_j, \ldots, 1, \ldots, u_k, \ldots, u_d)\\
&\geq B(u_1, \ldots, u_i u_j u_k, \ldots, 1, \ldots, 1, \ldots, u_d) \geq \ldots\\
&\geq B(1, \ldots, u_1 \cdots u_d, \ldots, 1)\\
&= u_1 \cdots u_d.
\end{aligned}
$$

Since a multivariate aging function satisfies (B1), $B \in \mathscr{A}_2^+$ is equivalent to $B \in \mathscr{A}_1^+$ for the case $d = 2$. However, in the d-dimensional case, $d \geq 3$, $\mathscr{A}_2^+$ is strictly included in $\mathscr{A}_1^+$, as it will be shown in Example 2.3.

In the following example, we consider the case of the so-called time-transformed exponential (TTE) models (see [3,29]). These models can be characterized as those multivariate survival functions admitting an Archimedean survival copula.

Example 2.2: Let B be a multivariate aging function that can be written in the form

$$
B(\mathbf{u}) = \psi^{-1}\left(\sum_{i=1}^{d} \psi(u_i)\right) \tag{2.6}
$$

for some strictly decreasing $\psi : \mathbb{I} \to \overline{\mathbb{R}}_+$ such that $\psi(0) = +\infty$ and $\psi(1) = 0$. This ψ is usually called *additive generator* of B. Such a B belongs to the class of the d-dimensional *strict triangular norms* (see [20]). In particular, B is also a copula (usually called *strict Archimedean copula*) when ψ^{-1} is d-completely monotone (see [24]). Now, for a semi-copula B of type (2.6), the following statements can be proved:

(i) $B \in \mathscr{A}_1^+$ if, and only if, $B \in \mathscr{A}_2^+$, and this happens when $\psi(uv) \leq \psi(u) + \psi(v)$ for all $u, v \in \mathbb{I}$.

(ii) $B \in \mathscr{A}_3^+$ if, and only if, ψ^{-1} is log-convex (see [9,11]).

Notice that the multivariate aging functions $B_{\Pi,\overline{G}}$ of (2.2) are of the form (2.6), with $\psi(t) = -\log(\overline{G}(-\log(t)))$.

We conclude this section by providing some examples clarifying the relations among the above-mentioned classes.

Example 2.3: Let $f : \mathbb{I} \to \mathbb{I}$ be the function given by

$$
f(t) = \begin{cases} et, & t \in [0, e^{-2}] \\ e^{-1}, & t \in]e^{-2}, e^{-1}] \\ t & t \in]e^{-1}, 1]. \end{cases}
$$

Let $C : \mathbb{I}^3 \to \mathbb{I}$ be given by $C(u_1, u_2, u_3) = u_{(1)} f(u_{(2)}) f(u_{(3)})$, where $u_{(1)}, u_{(2)}$, and $u_{(3)}$ denote the components of $\mathbf{u}$ rearranged in increasing order. Since $f(1) = 1, f$ is increasing, and $f(t)/t$ is decreasing on $]0,1]$, it follows that C is a copula (see [13, Thm. 3]). Actually, C is the survival copula of a random vector (X_1, X_2, X_3) having the stochastic representation $X_i = \max(Y_i, Z)$ $(i = 1, 2, 3)$, where Y_1, Y_2, Y_3, and Z are independent lifetimes. Roughly speaking, C is the survival copula of a random vector of independent lifetimes (Y_1, Y_2, Y_3) affected by a common shock Z (see also [12]).

If follows from [13] that C belongs to $\mathscr{A}_1^+$. However, $C \notin \mathscr{A}_2^+$. In fact, by taking $u_1 = e^{-(5/2)}$, $u_2 = e^{-(3/2)}$, and $u_3 = e^{-(1/2)}$, we have that $C(u_1, u_2, u_3) = e^{-4} < e^{-(7/2)} = C(u_1, u_2u_3, 1)$.

Example 2.4: Let B be the multivariate aging function of type (2.6), where $\psi : \mathbb{I} \to \overline{\mathbb{R}}_+$ is given by

$$\psi(t) = \begin{cases} -\log(t), & t \in]0, e^{-2-\varepsilon}] \\ -\dfrac{\varepsilon}{1+\varepsilon}(\log(t)+1)+2, & t \in]e^{-2-\varepsilon}, e^{-1}] \\ -2\log(t), & t \in]e^{-1}, 1], \end{cases}$$

with $\varepsilon \in]0, 1[$. Now, let us consider $g : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$, $g(t) = \psi(\exp(-t))$, given by

$$g(t) = \begin{cases} 2t, & t \in [0, 1] \\ \dfrac{\varepsilon}{1+\varepsilon}(t-1)+2, & t \in (1, 2+\varepsilon] \\ t, & t \in (2+\varepsilon, +\infty[. \end{cases}$$

Now, g is not concave and, hence, ψ^{-1} is not log-convex. Thus, in view of Example 2.2, $B \notin \mathscr{A}_3^+$. However, it can be shown that $\psi(uv) \le \psi(u) + \psi(v)$ for all $u, v \in \mathbb{I}$. From Example 2.2, it follows that $B \in \mathscr{A}_2^+$.

3. MULTIVARIATE AGING NOTIONS OF NBU AND IFR FOR EXCHANGEABLE RANDOM VARIABLES

In this section, the families $\mathscr{A}_1^+$, $\mathscr{A}_2^+$, and $\mathscr{A}_3^+$ will be used in order to define notions of positive aging in terms of the multivariate aging function B. Notice that since negative properties can be introduced and studied in a similar way, they will not be considered in detail.

First, we recall the following notions of univariate aging for a survival function $\overline{G}$:

- $\overline{G}$ is NBU if, and only if, for all $x, y \in \overline{\mathbb{R}}_+$, $\overline{G}(x+y) \le \overline{G}(x)\overline{G}(y)$.
- $\overline{G}$ is IFR if, and only if, $\overline{G}$ is log-concave.

As stated in Section 1, we aim at extending an aging notion from the univariate case to the d-dimensional case $(d \ge 2)$, following the line of [6]. To this end, we link univariate aging notions to analytical properties of a multivariate aging function. More precisely, we link properties of a survival function $\overline{G}$ to properties of the multivariate aging function $B_{\Pi,\overline{G}}$, which is associated with d i.i.d. lifetimes whose marginal survival function is $\overline{G}$. The following result can be given.

PROPOSITION 3.1: *Under Assumption 2, the following statements are equivalent:*

(a) $\overline{G}$ *is NBU.*

(b) $B_{\Pi,\overline{G}} \in \mathscr{A}_1^+$.

(c) $B_{\Pi,\overline{G}} \in \mathscr{A}_2^+$.

PROOF:

(a)$\Longleftrightarrow$(b): Let $\overline{G}$ be NBU. It can be proved by induction that for all $x, y \in \overline{\mathbb{R}}_+$, $\overline{G}(x+y) \le \overline{G}(x)\overline{G}(y)$ is equivalent to

$$\overline{G}\left(\sum_{i=1}^d x_i\right) \le \prod_{i=1}^d \overline{G}(x_i),$$

for any $\mathbf{x} \in \overline{\mathbb{R}}_+^d$. Setting $x_i = -\log(u_i)$, we obtain that for all $\mathbf{u} \in \mathbb{I}^d$,

$$\overline{G}(-\log(u_1 \cdots u_d)) \le \overline{G}(-\log(u_1)) \cdots \overline{G}(-\log(u_d)), \tag{3.1}$$

from which it straightforwardly follows that

$$\exp\Big(-\overline{G}^{-1}(\overline{G}(-\log(u_1 \cdots u_d)))\Big) \le \exp\Big(-\overline{G}^{-1}(\overline{G}(-\log(u_1)) \cdots \overline{G}(-\log(u_d)))\Big)$$

(i.e., $B_{\Pi,\overline{G}} \ge \Pi_d$).

(a)$\Longleftrightarrow$(c): Since $\overline{G}$ is NBU, $\overline{G}(-\log(u_iu_j)) \le \overline{G}(-\log(u_i))\overline{G}(-\log(u_j))$ holds for all $u_i, u_j \in \mathbb{I}$. By multiplying both the sides of the inequality by $\prod_{k\in\mathscr{I}} \overline{G}(-\log(u_k))$, where $\mathscr{I} = \{1, 2, \ldots, d\} \setminus \{i, j\}$ and $u_k \in \mathbb{I}$ for every $k \in \mathscr{I}$ and applying the function $\exp \circ (-\overline{G}^{-1})$ to both members, we obtain

$$B_{\Pi,\overline{G}}(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_d) \ge B_{\Pi,\overline{G}}(u_1, \ldots, u_iu_j, \ldots, 1, \ldots, u_d)$$

(i.e., $B_{\Pi,\overline{G}} \in \mathscr{A}_2^+$). ■

Therefore, we can write $\mathscr{A}_1^+ \cap \{B_{\Pi,\overline{G}} : \overline{G} \text{ is NBU}\} = \mathscr{A}_2^+ \cap \{B_{\Pi,\overline{G}} : \overline{G} \text{ is NBU}\}$. Notice that, in general, $\mathscr{A}_1^+ \neq \mathscr{A}_2^+$.

PROPOSITION 3.2: *Under Assumption 2, the following statements are equivalent:*

(a) $\overline{G}$ *is IFR.*

(b) $B_{\Pi,\overline{G}} \in \mathscr{A}_3^+$.

PROOF: Let $\overline{G}$ be IFR. As it easily follows, this fact is equivalent to

$$\frac{\overline{G}(x_i+\sigma)}{\overline{G}(x_i)} \geq \frac{\overline{G}(x_j+\sigma)}{\overline{G}(x_j)},$$

for any $x_i, x_j \in \mathbb{R}_+$, $x_i \leq x_j$ and $\sigma \geq 0$. By substituting $x_i = -\log(u_i)$, $x_j = -\log(u_j)$, and $\sigma = -\log(s)$, we obtain

$$\overline{G}(-\log(u_i s))\overline{G}(-\log(u_j)) \geq \overline{G}(-\log(u_j s))\overline{G}(-\log(u_i)),$$

for any $u_i, u_j \in (0,1]$, $u_i \geq u_j$ and $s \in (0,1)$. By multiplying both the sides of the inequality by $\prod_{k\in\mathscr{I}} \overline{G}(-\log(u_k))$, where $\mathscr{I} = \{1,2,\ldots,d\} \setminus \{i,j\}$ and $u_k \in \mathbb{I}$ for every $k, \in \mathscr{I}$ and applying the function $\exp \circ(-\overline{G}^{-1})$ to both the members, we obtain

$$B_{\Pi,\overline{G}}(u_1,\ldots,u_i s,\ldots,u_j,\ldots,u_d) \geq B_{\Pi,\overline{G}}(u_1,\ldots,u_i,\ldots,u_j s,\ldots,u_d)$$

(i.e., $B_{\Pi,\overline{G}} \in \mathscr{A}_3^+$). ■

The previous proposition is actually a reformulation in terms of the multivariate aging function of well-known results concerning the joint survival function $\overline{F} = \overline{F}_{\Pi,\overline{G}}$ of i.i.d. lifetimes that are IFR. As noted several times in the literature (see, e.g., [2,3,29]), such a $\overline{F}$ is *Schur-concave*; that is, for every $s \geq 0$ and for every $i,j \in \{1,2,\ldots,d\}$, $i<j$, the mapping

$$x_i \longmapsto \overline{F}(x_1,\ldots,x_{i-1},x_i,x_{i+1}\ldots,x_{j-1},s-x_i,x_{j+1},\ldots,x_d)$$

is decreasing on $[s/2,+\infty]$ (see [23, A.2.b]). This is equivalent to

$$\overline{F}(x_1,\ldots,x_i+\tau,\ldots,x_j-\tau,\ldots,x_d) \geq \overline{F}(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_d) \tag{3.2}$$

for all $i,j \in \{1,2,\ldots,d\}$, $i<j$, for every $\mathbf{x} \in \overline{\mathbb{R}}_+^d$ such that $x_i \leq x_j$, and for every $\tau \in [0, x_j - x_i]$.

Remark 3.3: As noted, $B_{\Pi,\overline{G}}$ is actually a d-dimensional strict triangular norm additively generated by $\psi = (-\log)\circ\overline{G}\circ(-\log)$. In this context, Propositions 3.1 and 3.2 can be reinterpreted in the following sense: Univariate aging properties of ψ^{-1}, which is a univariate survival function, reflect on special inequalities holding for the triangular norm generated by ψ. As a consequence, these results can be seen as extensions of the investigations in [1].

Now, by using Propositions 3.1 and 3.2 and the scheme presented in Section 1, we introduce the following notions of multivariate aging for an exchangeable survival function $\overline{F}$.

DEFINITION 3.4: *Under Assumption 1, we say the following:*

- $\overline{F}$ *is* B-multivariate-NBU of the first type *(B-MNBU1) if, and only if,* $B_{\overline{F}} \in \mathscr{A}_1^+$.
- $\overline{F}$ *is* B-multivariate-NBU of the second type *(B-MNBU2) if, and only if,* $B_{\overline{F}} \in \mathscr{A}_2^+$.
- $\overline{F}$ *is* B-multivariate-IFR *(B-MIFR) if, and only if,* $B_{\overline{F}} \in \mathscr{A}_3^+$.

In order to avoid confusion with other multivariate notions of aging introduced in the previous literature, we used the prefix B for the notions introduced above. This also serves to underline the fact that all of these notions are expressed in terms of the multivariate aging functions B of $\overline{F}$. Now, we would like to investigate some properties of these notions.

First, notice that any k-dimensional marginal of $\overline{F}$ $(2 \leq k \leq d-1)$ has the same multivariate aging property of $\overline{F}$. This point is formalized in the following result.

PROPOSITION 3.5: *Suppose that Assumption 1 holds. For every $2 \leq k \leq d$, let $\overline{F}^{(k)}$ be the k-dimensional marginal of $\overline{F}$. If $\overline{F}$ is B-MNBU1 (respectively, B-MNBU2 or B-MIFR), then $\overline{F}^{(k)}$ is B-MNBU1 (respectively, B-MNBU2 or B-MIFR).*

PROOF: If $\overline{F}^{(k)} : \overline{\mathbb{R}}_+^k \to \mathbb{I}$ is the k-dimensional margin of $\overline{F}$ $(2 \leq k \leq d)$, given by

$$\overline{F}^{(k)}(x_1, \ldots, x_k) = \overline{F}(x_1, \ldots, x_k, 0, \ldots, 0),$$

then it follows from (2.1) that

$$B_{\overline{F}^{(k)}}(u_1, \ldots, u_k) = B_{\overline{F}}(u_1, \ldots, u_k, 1, \ldots, 1).$$

Easy calculations show that $B_{\overline{F}^{(k)}}$ is in $\mathscr{A}_1^+$ (respectively, $\mathscr{A}_2^+$ or $\mathscr{A}_3^+$) when $B_{\overline{F}}$ is in $\mathscr{A}_1^+$ (respectively, $\mathscr{A}_2^+$ or $\mathscr{A}_3^+$), which is the desired assertion. ■

The previous definitions of multivariate aging admit some probabilistic interpretations in terms of conditional survival probabilities for residual lifetimes. Before stating them, we clarify the notation. For every $\mathbf{x} \in \overline{\mathbb{R}}_+^d$ we denote by $\hat{\mathbf{x}}_i$ the vector of $\overline{\mathbb{R}}_+^{d-1}$ obtained by depriving $\mathbf{x}$ of its ith component. Similar agreement will be applied to random vectors.

PROPOSITION 3.6: *Under Assumption 1, the following statements hold:*

(a) $\overline{F}$ *is B-MNBU1 if, and only if, for every $i \in \{1, 2, \ldots, d\}$, $\mathbf{x} \in \overline{\mathbb{R}}_+^d$ and $\tau > 0$,*

$$\mathbb{P}(X_1 > x_1, \ldots, X_i > x_i + \tau, \ldots X_d > x_d \mid X_i > x_i) \geq \mathbb{P}(X_i > x_1 + \ldots + x_i + \ldots + x_d + \tau \mid X_i > x_i). \qquad \textbf{(3.3)}$$

(b) $\overline{F}$ *is B-MNBU2 if, and only if, for all* $i, j \in \{1, 2, \dots, d\}$, $i \neq j$, *for every* $\hat{\mathbf{x}}_j \in \overline{\mathbb{R}}_+^{d-1}$ *and* $\tau > 0$,

$$\mathbb{P}(X_j > \tau \mid \hat{\mathbf{X}}_j > \hat{\mathbf{x}}_j) \geq \mathbb{P}(X_i > \tau + x_i \mid \hat{\mathbf{X}}_j > \hat{\mathbf{x}}_j). \quad \textbf{(3.4)}$$

(c) $\overline{F}$ *is B-MIFR if, and only if, for all* $i, j \in \{1, 2, \dots, d\}$, *for every* $\mathbf{x} \in \overline{\mathbb{R}}_+^d$ *such that* $x_i \leq x_j$, *and for every* $\tau > 0$,

$$\mathbb{P}(X_i > x_i + \tau \mid \mathbf{X} > \mathbf{x}) \geq \mathbb{P}(X_j > x_j + \tau \mid \mathbf{X} > \mathbf{x}). \quad \textbf{(3.5)}$$

PROOF:

(a) By definition, $\overline{F}$ is B-MNBU1 if, and only if, for every $\mathbf{u} \in \mathbb{I}^d$,

$$\exp\left(\overline{G}^{-1}(\overline{F}(-\log(u_1), \dots, -\log(u_d)))\right) \geq u_1 \cdots u_d.$$

Thus, for every $\mathbf{x} \in \overline{\mathbb{R}}_+^d$, we have that

$$\overline{F}(x_1, \dots, x_d) \geq \overline{G}(x_1 + \cdots + x_d), \quad \textbf{(3.6)}$$

which is equivalent to the fact that Eq. (3.3) holds.

(b) Since $\overline{F}$ is B-MNBU2, for all $i, j \in \{1, \dots, d\}$, $i \neq j$, and for every $\mathbf{u} \in \mathbb{I}^d$,

$$\exp\left(\overline{G}^{-1}(\overline{F}(-\log(u_1), \dots, -\log(u_i), \dots, -\log(u_j), \dots, -\log(u_d)))\right)$$
$$\geq \exp\left(\overline{G}^{-1}(\overline{F}(-\log(u_1), \dots, -\log(u_i u_j), \dots, 1, \dots, -\log(u_d)))\right),$$

which is equivalent to

$$\overline{F}(x_1, \dots, x_i, \dots, \tau, \dots, x_d) \geq \overline{F}(x_1, \dots, \tau + x_i, \dots, 0, \dots, x_d) \quad \textbf{(3.7)}$$

for all $\hat{\mathbf{x}}_i \in \overline{\mathbb{R}}_+^{d-1}$ and $\tau > 0$. This last condition can be expressed as

$$\mathbb{P}(X_j > \tau \mid X_1 > x_1, \dots, X_i > x_i, \dots, X_j > 0, \dots, X_d > x_d)$$
$$\geq \mathbb{P}(X_i > \tau + x_i \mid X_1 > x_1, \dots, X_i > x_i, \dots, X_j > 0, \dots, X_d > x_d),$$

which is the desired assertion.

(c) We just have to prove that $\overline{F}$ is B-MIFR if, and only if, $\overline{F}$ is Schur-concave. Then the assertion will follow, since the Schur-concavity of $\overline{F}$ is equivalent to the fact that Eq. (3.5) holds (see [28,29, Prop. 4.15]).

Now, the equivalence between $\overline{F}$ being B-MIFR and $\overline{F}$ being Schur-concave follows by extending [9, Lemma 4.2] from the bivariate to the d-dimensional case. In detail, $\overline{F}$ is Schur-concave if, and only if, for all $i, j \in$

$\{1, 2, \ldots, d\}$, $i < j$, for every $\mathbf{x} \in \overline{\mathbb{R}}_+^d$ such that $x_i \leq x_j$, and for every $\tau \in [0, (x_j - x_i)/2]$, inequality (3.2) holds. In terms of $B_{\overline{F}}$, this is equivalent to

$$B_{\overline{F}}(e^{-x_1}, \ldots, e^{-x_i-\tau}, \ldots, e^{-x_j+\tau}, \ldots, e^{-x_d}) \geq B_{\overline{F}}(e^{-x_1}, \ldots, e^{-x_i}, \ldots, e^{-x_j}, \ldots, e^{-x_d}). \quad \textbf{(3.8)}$$

In other words,

$$B_{\overline{F}}(u_1, \ldots, u_i s, \ldots, u_j/s, \ldots, u_d) \geq B_{\overline{F}}(u_1, \ldots, u_d), \quad \textbf{(3.9)}$$

for all $i, j \in \{1, 2, \ldots, d\}$, $i < j$, for every $\mathbf{u} \in]0, 1]^d$ such that $u_i \geq u_j$, and for every $s \in [u_j/u_i, 1]$, which is an equivalent way of expressing the fact that $B_{\overline{F}} \in \mathscr{A}_3^+$. ■

Note that conditions (3.4) and (3.5) can be expressed as comparisons between residual lifetimes, conditionally on a same history. Specifically, the condition $\overline{F}$ being B-MNBU2 is equivalent to

$$\left[X_i \mid \hat{\mathbf{X}}_i > \hat{\mathbf{x}}_i\right] \geq_{\text{st}} \left[X_j - x_j \mid \hat{\mathbf{X}}_i > \hat{\mathbf{x}}_i\right], \quad \textbf{(3.10)}$$

for all $i, j \in \{1, 2, \ldots, d\}$, $i \neq j$, and for every $\mathbf{x} \in \overline{\mathbb{R}}_+^d$, where $\geq_{\text{st}}$ denotes the univariate usual stochastic order (see [26]). Instead, the fact that $\overline{F}$ is B-MIFR can be expressed as

$$[X_i - x_i \mid \mathbf{X} > \mathbf{x}] \geq_{\text{st}} [X_j - x_j \mid \mathbf{X} > \mathbf{x}], \quad \textbf{(3.11)}$$

for all $i, j \in \{1, 2, \ldots, d\}$, for every $\mathbf{x} \in \overline{\mathbb{R}}_+^d$ such that $x_i \leq x_j$.

Comparisons of laws of different lifetimes, conditionally on the same state of information, have been considered in [4,5] as a way for defining possible notions of multivariate aging that are appropriate in situations where "the (Bayesian) dependence due to learning about some unobservable quantity cannot be neglected" (see [4]). Notions of multivariate aging introduced in this way are different from the ones introduced by comparing the laws of the same vector of surviving components, conditional on two different states of information (see, e.g., [22,26] and the references therein).

Thanks to the probabilistic interpretations given by (3.10) and (3.11), an interesting link between B-MNBU2 and B-MIFR can be proved. Let us consider the vector of the residual lifetimes of $\mathbf{X}$ at time $t > 0$, $\mathbf{X}_t = [\mathbf{X} - \mathbf{t} \mid \mathbf{X} > \mathbf{t}]$, where $\mathbf{t} = (t, \ldots, t)$. Let us denote by $\overline{F}_t : \overline{\mathbb{R}}_+^d \to \mathbb{I}$ the joint survival function of $\mathbf{X}_t$ and by $B_{\overline{F}_t}$ the corresponding multivariate aging function. By extending some results related to the bivariate case (see [7–9,18]), the following one can be proved.

Proposition 3.7: *Under Assumption 1, for every $t \geq 0$, $\overline{F}_t$ is B-MNBU2 if, and only if, $\overline{F}$ is B-MIFR.*

PROOF: $\overline{F}_t$ is B-MNBU2 for every $t \geq 0$ if, and only if,

$$\overline{F}(x_1 + t, \ldots, x_i + t, \ldots, x_j + t, \ldots, x_d + t)$$
$$\geq \overline{F}(x_1 + t, \ldots, t, \ldots, x_j + x_i + t, \ldots, x_d + t)$$

for every $t \geq 0$, $\mathbf{x} \in \overline{\mathbb{R}}_+^d$ and $i, j \in \{1, 2, \ldots, d\}$, which is equivalent to the fact that $\overline{F}$ is Schur-concave. ■

Note that if $\overline{F}$ is B-MNBU2, then $\overline{F}_t$ might not be B-MNBU2 for some $t > 0$ (see [17] for an example in the bivariate case). However, for the notion of B-MIFR, we can prove the following result.

COROLLARY 3.8: *Under Assumption 1, if $\overline{F}$ is B-MIFR, then $\overline{F}_t$ is B-MIFR for every $t \geq 0$.*

PROOF: From Proposition 3.7, if $\overline{F}$ is B-MIFR, then $\overline{F}_{t+s}$ is B-MNBU2 for every $t, s \geq 0$. As a consequence, $\overline{F}_t$ is B-MIFR for every $t \geq 0$. ■

Concerning inequality (3.3), it is not clear whether it can also be expressed as comparison of lifetimes conditionally on the same history, in a similar way to the inequalities in (3.10) and (3.11). However, it is possible to give it an intuitive interpretation in reliabilistic terms, similarly to what was done in [9, Example 4.2] for the case $d = 2$.

Remark 3.9: Notice that inequality (3.4) implies inequality (3.3); this can be seen from Section 2 by using the multivariate aging function $B_{\overline{F}}$ and the given definitions of B-MNBU1 and B-MNBU2. Actually, as shown in Example (2.2), inequalities (3.4) and (3.3) coincide for TTE models but not in general. Consider, for instance, a multivariate survival function $\overline{F}$ whose marginals are exponential and whose copula is the one of Example 2.3.

The notions of multivariate aging introduced in Definition 3.4 are preserved under mixtures, as specified by the following proposition.

PROPOSITION 3.10: *Let $(\overline{F}_\theta)_{\theta \in \Theta}$ be a family of survival functions satisfying Assumption* 1. *Let λ be a distribution on Θ. Let $\overline{F}$ be the mixture of $(\overline{F}_\theta)_{\theta \in \Theta}$ with respect to λ, given, for every $\mathbf{x} \in \overline{\mathbb{R}}_+^d$, by*

$$\overline{F}(\mathbf{x}) = \int_\Theta \overline{F}_\theta(\mathbf{x})\, d\lambda(\theta).$$

The following statements hold:

(a) If $\overline{F}_\theta$ is B-MNBU1 for every $\theta \in \Theta$, then $\overline{F}$ is B-MNBU1.

(b) *If $\overline{F}_\theta$ is B-MNBU2 for every $\theta \in \Theta$, then $\overline{F}$ is B-MNBU2.*

(c) *If $\overline{F}_\theta$ is B-MIFR for every $\theta \in \Theta$, then $\overline{F}$ is B-MIFR.*

PROOF: Part (a) follows by considering that every $\overline{F}_\theta$ satisfies (3.6) and hence the mixture $\overline{F}$ satisfies (3.6), which is an equivalent formulation of the B-MNBU1 property for $\overline{F}$. Analogously, part (b) easily follows from the fact that every $\overline{F}_\theta$ satisfies (3.7).

Finally, if every $\overline{F}_\theta$ is B-MIFR, then it is Schur-concave. As a consequence, the mixture $\overline{F}$ is also Schur-concave and, therefore, B-MIFR (see [23]). ■

Consequently, the following result can be easily derived.

PROPOSITION 3.11: *Under Assumption 1, suppose that $\overline{F}$ is the survival function of conditionally i.i.d. lifetimes given a common factor Θ with prior distribution λ. Moreover, suppose that $\overline{G}(\cdot \mid \theta)$ is NBU (respectively, IFR). Then $\overline{F}$ is B-MNBU2 (respectively, B-MIFR).*

Thus, the given definitions of multivariate aging have an interesting property: Mixtures of i.i.d. lifetimes that are NBU (respectively, IFR) conditionally on the same factor Θ are also multivariate NBU (respectively, IFR).

4. DISCUSSION AND CONCLUDING REMARKS

In this article, we have presented an extension to the d-dimensional case ($d \geq 2$) of bivariate aging notions discussed in [6,9].

An interesting point concerns the extension of the NBU property, that, for the multivariate case $d \geq 3$, can lead us to two different formulations. In fact, this happens when the components of $\overline{F}$ are coupled by a copula $K_{\overline{F}}$ outside the Archimedean class (see Example 2.2).

Here, it should be considered that, following the scheme (i)–(iii) presented in Section 1, it is possible that several properties of $B_{\Pi,\overline{G}}$ describe the same bivariate aging of a joint survival function $\overline{F}_{\Pi,\overline{G}}$ of independent components. In [9], for instance, different notions of bivariate IFR have been discussed. When such situations occur, it is quite natural to consider all of these different multivariate notions of a given univariate aging property P and select among them those properties of $B_{\Pi,\overline{G}}$ with some interesting probabilistic meaning.

Also for these reasons, we wanted to stress that in this article the introduced multivariate aging notions exhibit some particular features: they are closed under mixtures and can be characterized in terms of comparisons of conditional survival functions given a same observed history.

Finally, we would like to discuss a possible application of our results to the construction of multivariate stochastic models. To this end, we present the following proposition, which extends some results in [9] to the multivariate case.

PROPOSITION 4.1: *Under Assumption 1, the following statements hold:*

(a) *If $K \in \mathscr{A}_1^+$ and $\overline{G}$ is NBU, then $\overline{F}$ is B-MNBU1.*
(b) *If $K \in \mathscr{A}_2^+$ and $\overline{G}$ is NBU, then $\overline{F}$ is B-MNBU2.*
(c) *If $K \in \mathscr{A}_3^+$ and $\overline{G}$ is IFR, then $\overline{F}$ is B-MIFR.*

PROOF:

(a) Let $K \in \mathscr{A}_1^+$. Then, for every $\mathbf{u} \in \mathbb{I}^d$,

$$\exp\Big(-\overline{G}^{-1}\left(K(\overline{G}(-\log(u_1)),\ldots,\overline{G}(-\log(u_d)))\right)\Big) \geq B_{\Pi,\overline{G}}(u_1,\ldots,u_d).$$

By considering (2.1) and Proposition 3.1(a), it follows that $B \in \mathscr{A}_1^+$.

(b) Let $K \in \mathscr{A}_2^+$. Then, for every $\mathbf{u} \in \mathbb{I}^d$,

$$\begin{aligned}
&\overline{G}\left(-\log\Big(B\Big(e^{-\overline{G}^{-1}(u_1)},\ldots,e^{-\overline{G}^{-1}(u_i)},\ldots,e^{-\overline{G}^{-1}(u_j)},\ldots,e^{-\overline{G}^{-1}(u_d)}\Big)\Big)\right)\\
&\quad\geq \overline{G}\left(-\log\Big(B\Big(e^{-\overline{G}^{-1}(u_1)},\ldots,e^{-\overline{G}^{-1}(u_iu_j)},\ldots,1,\ldots,e^{-\overline{G}^{-1}(u_d)}\Big)\Big)\right)\\
&\quad\geq \overline{G}\left(-\log\Big(B\Big(e^{-\overline{G}^{-1}(u_1)},\ldots,e^{-(\overline{G}^{-1}(u_i)+\overline{G}^{-1}(u_j))},\ldots,1,\ldots,e^{-\overline{G}^{-1}(u_d)}\Big)\Big)\right),
\end{aligned}$$

where the last inequality follows from the fact that $\overline{G}$ is NBU. Setting $x_i = e^{-\overline{G}^{-1}(u_i)}$, it follows that $B \in \mathscr{A}_2^+$, which is the desired assertion.

(c) Since $K \in \mathscr{A}_3^+$, for every $\mathbf{u} \in \mathbb{I}^d$ such that $u_i \geq u_j$ and for every $s \in (0,1)$,

$$K(u_1,\ldots,u_is,\ldots,u_j,\ldots,u_d) \geq K(u_1,\ldots,u_i,\ldots,u_js,\ldots,u_d).$$

In particular, for every $0 < s_j \leq s_i < 1$,

$$K(u_1,\ldots,u_is_i,\ldots,u_j,\ldots,u_d) \geq K(u_1,\ldots,u_i,\ldots,u_js_j,\ldots,u_d). \qquad \textbf{(4.1)}$$

Now, for every $k \in \{1,2,\ldots,d\}$, set

$$\alpha_k = \overline{G}^{-1}(u_k), \qquad s_i = \frac{\overline{G}(\alpha_i+\sigma)}{\overline{G}(\alpha_i)}, \qquad s_j = \frac{\overline{G}(\alpha_j+\sigma)}{\overline{G}(\alpha_j)},$$

where $\sigma = \overline{G}^{-1}(u_is_i) - \overline{G}^{-1}(u_i) = \overline{G}^{-1}(u_js_j) - \overline{G}^{-1}(u_j)$. Since $\overline{G}$ is IFR, $s_i \geq s_j$. Moreover, from (4.1) we obtain

$$\begin{aligned}
&K\left(\overline{G}(\alpha_1),\ldots,\overline{G}(\alpha_i+\sigma),\ldots,\overline{G}(\alpha_j),\ldots,\overline{G}(\alpha_d)\right)\\
&\qquad\geq K\left(\overline{G}(\alpha_1),\ldots,\overline{G}(\alpha_i),\ldots,\overline{G}(\alpha_j+\sigma),\ldots,\overline{G}(\alpha_d)\right).
\end{aligned}$$

By applying to both the sides of this inequality the transformation $\exp \circ (-\overline{G}^{-1})$, we have

$$B(x_1, \ldots, x_i s', \ldots, x_j, \ldots, x_d) \geq B(x_1, \ldots, x_i, \ldots, x_j s', \ldots, x_d),$$

for every $\mathbf{x} \in \mathbb{I}^d$ such that $x_i \geq x_j$ and for every $s' \in (0, 1)$, that is $B_{\overline{F}} \in \mathscr{A}_3^+$. ■

Remark 4.2: As already noted, both the conditions $K \in \mathscr{A}_2^+$ and $K \in \mathscr{A}_3^+$ imply that $K \in \mathscr{A}_1^+$, which is considered a notion of multivariate positive dependence. Thus, roughly speaking, Proposition 4.1 suggests that positive univariate aging and (some kind of) positive dependence play in favor of positive multivariate aging. However, note that positive multivariate aging can coexist with several forms of dependence and univariate aging. This fact was already stressed, for example, in [9].

Interestingly (at least for some statistical purposes), Proposition 4.1 can be used when one wants to construct, for components judged to be similar, a multivariate survival model satisfying some kind of aging condition. In fact, by using the celebrated Sklar's theorem [27], such a model can be constructed just by conveniently choosing some univariate survival function $\overline{G}$ (e.g., satisfying NBU or IFR property) and a suitable copula K (belonging to some class $\mathscr{A}_i^+$); hence, we join them in order to obtain the multivariate survival function $\overline{F} = K(\overline{G}, \ldots, \overline{G})$. Hence, this procedure provides sufficient conditions for multivariate aging in terms of univariate aging and stochastic dependence.

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