January 2005

Risk-Aversion and the Investment-Uncertainty Relationship: A Comment

Risk-aversion and the investment-uncertainty relationship: a comment

Enrico Saltari and Davide Ticchi*

Abstract

This paper shows that the solution of Nakamura's (1999) model is incorrect. We propose an alternative framework that allows us to obtain closed form results on the investment-uncertainty relationship.

JEL classification: D81; D92; E22

Keywords: Investment; Uncertainty; Risk aversion; Intertemporal substitution

* Enrico Saltari: Department of Public Economics, University of Rome "La Sapienza", Italy. Davide Ticchi (corresponding author): Department of Economics, University of Urbino, Italy and Department of Economics, University Pompeu Fabra, Spain. Address: Facoltà di Economia, via Saffi 42, 61029 Urbino, Italy. E-mail: ticchi@econ.uniurb.it. Tel.: +39-0722305556; fax: +39-0722305550. We are grateful to Antonio Ciccone for comments and suggestions. Ticchi gratefully acknowledges financial support from the European Commission through the RTN grant "Specialization Versus Diversification." Remaining errors are all ours.

Risk-aversion and the investment-uncertainty relationship: a comment

In a recent paper Nakamura (1999) tries to explain the negative relationship between investment and uncertainty obtained in many empirical studies by assuming that firms are risk averse. The model is constructed using the framework of Jorgenson (1963) with uncertainty. The competitive firm produces a good with a constant returns to scale Cobb-Douglas production function and its instantaneous utility function exhibits constant relative risk aversion. Nakamura obtains a negative relationship between investment and uncertainty whenever the coefficient of relative risk aversion is higher than the elasticity of output with respect to labor but less than one.

In this short comment we want to highlight a mistake in Nakamura's results concerning the value function used to derive the investment function. The guess of the value function (equation (A.1), p. 361) turns out not to be verified because the undetermined coefficient of the guess (equation (A.8), p. 362) is not a constant but depends on the current value of the stochastic variable (the output price), which changes over time. As the investment function is incorrect, the comparative static results in Nakamura (1999) are invalid.

The output price in Nakamura's model follows a geometric Brownian motion (equation (3), p. 358). It may be worthwhile emphasizing that problems where the stochastic variable follows a geometric Brownian motion and individuals have a constant relative risk aversion utility function have no closed form solution. For a detailed discussion of the issues that arise in such frameworks and the possible solutions see, for example, Merton (1971). The only case where there is a closed form solution to Nakamura's model is when the utility function is logarithmic. However, uncertainty has no effect on investment in this case because the income and substitution effects exactly offset each other.

Nevertheless, the idea of analyzing the relationship between investment and uncertainty in a dynamic model with risk averse firms as proposed by Nakamura is interesting. We therefore propose an alternative framework that allows us to obtain closed form results on the investment-uncertainty relationship. Assume that the technology of the competitive firm is described by a constant return to scale Cobb-Douglas production function $Y_t = K_t^{1-\alpha} L_t^{\alpha}$, where K_t is the stock of capital and L_t is the amount of labor employed. Suppose also that the output price is given by $p_t = \exp\{\vartheta_t\}$ where ϑ_t is an *i.i.d.* random variable distributed according to $N\left(\overline{\vartheta} - \frac{1}{2}\sigma^2, \sigma^2\right)$ and where $\overline{\vartheta}$ and σ^2 are constant over time. This parameterization implies that an increase in σ^2 increases the variance of output price without affecting the mean value of p_t. Then, the firm's cash flow at time t is given by

$$\pi_t = p_t K_t^{1-\alpha} L_t^{\alpha} - wL_t - qI_t \tag{1}$$

where w is the constant real wage and q is the price of capital, which we assume to be constant and equal to one. In each period, the firm first observes the realization of the shock and then adjust the amount of labor. Therefore, the optimal amount of labor is $L_t^* = (\alpha / w)^{1/(1-\alpha)} p_t^{1/(1-\alpha)} K_t$ and the firm's cash flow simplifies to

$$\pi_t = B_t^{\eta} K_t - I_t \tag{2}$$

where $B_t = Bp_t$, $B = (1-\alpha)^{1-\alpha}(\alpha / w)^{\alpha}$ and $\eta = 1/(1-\alpha)$. It is clear that B_t^{η} is equal to the marginal revenue product of capital and $B_t^{\eta} K_t$ is equal to operating profits.

Assume that the firm maximizes the intertemporal cash flows with the following expected (constant relative risk aversion) utility preferences

$$V_t(K_t, \vartheta_t) = \max_{\{I_t\}} \left\{ \frac{\pi_t^{1-\gamma} - 1}{1-\gamma} + \beta E_t V_{t+1}(K_{t+1}, \vartheta_{t+1}) \right\} \tag{3}$$

subject to the capital accumulation equation

$$K_{t+1} = (1-\delta) K_t + I_t \tag{4}$$

where δ is the rate of capital depreciation, $\beta \in (0,1)$ is the utility discount factor and $\gamma > 0$ is the coefficient of relative risk aversion. The main difference between this framework and the one of Nakamura is that we consider a discrete time model with identically, independently distributed output price shocks while his model is in continuous time with the output price following a Brownian motion.[1]

Our guess for the value function of the maximization problem (3) subject to (4) is

$$V_t = \frac{\xi^{1-\gamma} \left[B_t^{\eta} + (1-\delta)\right]^{1-\gamma} K_t^{1-\gamma} - 1}{(1-\beta)(1-\gamma)} \tag{5}$$

with the investment function given by

$$I_t = B_t^{\eta} K_t - \lambda \left[B_t^{\eta} + (1-\delta)\right] K_t \tag{6}$$

where ξ and λ are constants to be determined. Solving the maximization problem we obtain that

$$\lambda = 1 - \beta^{\frac{1}{\gamma}} \left\{ E_t \left[B_{t+1}^{\eta} + (1-\delta)\right]^{1-\gamma} \right\}^{\frac{1}{\gamma}} \tag{7}$$

and $\xi = \left[(1-\beta)\lambda^{-\gamma}\right]^{\frac{1}{1-\gamma}}$.

The effect of uncertainty on investment can be obtained by differentiating the investment function in (6) with respect to the volatility σ^2 of the output price shock. However, it is possible to derive a closed form solution of this derivative only when capital fully depreciates after production ($\delta = 1$). In this case,

[1] For a more general framework see Saltari and Ticchi (2000).

$$\lambda = 1-\beta^{\frac{1}{\gamma}}\left\{E_t\left[B_{t+1}^{\eta(1-\gamma)}\right]\right\}^{\frac{1}{\gamma}} = 1-\beta^{\frac{1}{\gamma}} B^{\eta\frac{1-\gamma}{\gamma}}\left[\exp\left\{\eta\bar{\vartheta}+\frac{1}{2}(\alpha-\gamma)\eta^2\sigma^2\right\}\right]^{\frac{1-\gamma}{\gamma}} \tag{8}$$

and

$$\frac{dI}{d\sigma^2} = -\frac{d\lambda}{d\sigma^2}B_t^{\eta}K_t = \frac{1}{2}\beta^{\frac{1}{\gamma}}\eta^2\frac{(1-\gamma)(\alpha-\gamma)}{\gamma}\left[\exp\left\{\eta\bar{\vartheta}+\frac{1}{2}(\alpha-\gamma)\eta^2\sigma^2\right\}\right]^{\frac{1-\gamma}{\gamma}} B_t^{\eta}K_t \tag{9}$$

which means that $sign(dI/d\sigma^2) = sign[(1-\gamma)(\alpha-\gamma)]$. It is easy to see that the effect of uncertainty on investment is generally positive except when $\alpha<\gamma<1$. Maybe surprisingly, this is the same result obtained by Nakamura. The explanation of this result however is different than the one suggested by Nakamura.

Our interpretation of this result is the following. The investment function in (6) (which is $I_t = (1-\lambda)B_t^{\eta}K_t$ if $\delta = 1$) implies that the entrepreneur always invest a fraction $1-\lambda$ of operating profits and consume a fraction λ. Hence, the key distribution coefficient λ can be interpreted as the "marginal propensity to consume" out of operating profits. When λ increases the entrepreneur consumes more and invests less and vice-versa. This coefficient can be rewritten as

$$\lambda = 1-\beta^{\frac{1}{\gamma}}\Re^{\frac{1-\gamma}{\gamma}} = 1-\beta^{\varepsilon}\Re^{\varepsilon-1} \tag{10}$$

where $\varepsilon = 1/\gamma$ is the elasticity of intertemporal substitution and

$$\Re \equiv \left\{E_t\left[B_{t+1}^{\eta(1-\gamma)}\right]\right\}^{\frac{1}{1-\gamma}} = B^{\eta}\exp\left\{\eta\bar{\vartheta}+\frac{1}{2}(\alpha-\gamma)\eta^2\sigma^2\right\} \tag{11}$$

is the certainty equivalent of the marginal revenue product of capital. Given the assumption of full capital depreciation, B_{t+1}^{η} is also the return to capital because it is what the owner of the firm receives at time $t+1$ if he consumes one unit less at time t and invests it in capital. Therefore, $\Re$ also represents the certainty equivalent of the return to capital and from (11) it is clear that $\Re$ is increasing in the variance σ^2 of the output price shock if $\alpha>\gamma$ and decreasing when $\alpha<\gamma$. Indeed, when uncertainty increases there are two effects at work. The first, that we call *flexibility effect*, comes from the fact that labor is a flexible factor and the firm can substitute labor for capital after observing the realization of the output price shock. This implies that the marginal revenue product of capital is convex with respect to the price shock and, by Jensen's inequality, the expected marginal revenue product of capital is increasing in the volatility of the shock. The strength of this effect is positively related to the elasticity of output with respect to labor α. The second effect can be called *risk aversion effect* as it is generated by the entrepreneur's risk aversion. In choosing the level of investment the entrepreneur does not consider the expected return to capital but the correspondent certainty equivalent. The latter is negatively related to the riskiness of the return to capital which is a positive function of the variance σ^2 of the output price shock. The magnitude of this effect is positively correlated with the degree of the entrepreneur's risk aversion γ. The final effect of uncertainty on the certainty equivalent of the return to capital $\Re$ is ambiguous. When the risk aversion is small enough ($\gamma < \alpha$), the *flexibility effect* prevails on the *risk aversion effect*, and $\Re$ is increasing in the variance σ^2 of the output price shock. The opposite is true when the risk aversion is relatively big ($\gamma > \alpha$).

The variation of the certainty equivalent of the return to capital generated by an increase in uncertainty gives rise to an income and a substitution effect that affect the level of investment in opposite directions. The relative strengths of these two effects, and therefore the final effect of uncertainty on investment, depends on the magnitude of the elasticity of intertemporal substitution ε. If $\alpha < \gamma < 1$, then greater uncertainty reduces the certainty equivalent of the return to capital $\Re$. The reduction of $\Re$ generates a substitution effect that lowers investment because capital is less productive and an income effect that increases investment because the lower productivity of capital leads the entrepreneur to feel poorer. In this case the elasticity of intertemporal substitution is greater than one ($\varepsilon = 1/\gamma > 1$) implying that the substitution effect prevails over the income effect and leads to lower investment. Conversely, when risk aversion is sufficiently small ($0<\gamma<\alpha$), then more uncertainty increases the certainty equivalent of the return to capital $\Re$ and this raises investment. When γ is bigger than one greater uncertainty decreases the certainty equivalent of the return to capital $\Re$ (as $\gamma>\alpha$). The fact that the elasticity of intertemporal substitution is less than one ($\varepsilon = 1/\gamma < 1$) implies that the income effect more than balances the substitution effect and leads to more investment.

Uncertainty has no effect on investment when $\gamma=\alpha$ and when $\gamma=1$. In the first case a variation in the output price volatility does not change $\Re$ because the *flexibility effect* and the *risk aversion effect* exactly compensate each other; $\gamma=1$ implies that the utility function is logarithmic ($\varepsilon=1$) and the income and the substitution effect exactly offset each other.

References

Jorgenson, D.W., 1963. Capital theory and investment behavior. American Economic Review, Papers and Proceedings 53, 247–259.

Merton, R.C., 1971. Optimum consumption and portfolio rules. Journal of Economic Theory 3, 373–413.

Nakamura, T., 1999. Risk-aversion and the uncertainty-investment relationship: a note. Journal of Economic Behavior and Organization 38, 357–363.

Saltari, E., Ticchi, D., 2000. Can risk aversion really explain the negative investment-uncertainty relationship? Mimeo, University of Urbino, Italy.