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Convergence results for multivariate martingales

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Abstract

We present a new version of the Central Limit Theorem for multivariate martingales.

Keywords: Continuous-time multivariate martingales; Stable convergence

1. Introduction

In the literature there are several versions of the central limit theorem for multivariate martingales. The first version was given by Hutton and Nelson [8,9]. However, their assumption that the quadratic variation matrix converges when it is normalized by a scalar turned out to be rather restrictive for applications. Thus, some years later, Sørensen [15] published a central limit theorem in which the quadratic variation matrix converges when it is normalized by a diagonal matrix. The proof given by Sørensen is based on a result by Feigin [5, Theorem 2]. A result similar to that of Sørensen appears in Heyde [7]. The proof given by Heyde follows the one of Hutton and Nelson and it is based on an adaption of Theorem 3.2 in [6]. However, these theorems also turned out to be too restrictive for applications and so Küchler and Sørensen [12] gave a central limit theorem for multivariate martingales where the quadratic variation matrix is assumed to converge when it is normalized

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by a suitable full matrix. Once again, for the proof of this result, Theorem 2 of [5], is used. Finally, Theorem A.7.7 in [11] slightly extends the previous result.

If, for a given random variable U with values in the space of positive semi-definite $d \times d$-matrices, we denote by $\mathcal{N}(0, U)$ the corresponding *Gaussian kernel*, i.e. the family of Gaussian distributions $(\mathcal{N}(0, U(\omega)))_{\omega\in\Omega}$, then the theorem by Küchler and Sørensen [11] can be so stated:

Theorem 1.1. *On a probability space* $(\Omega, \mathcal{A}, P)$, *endowed with a filtration* $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$ *which satisfies the usual conditions, let* $M = (M_t)_{t\geqslant 0}$ *be a (right-continuous with limits from the left) d-dimensional martingale such that* $M_t^i \in L^2$ *for each t and i. Let us denote by Q the process* $[M, M]$. *Further let* $(a_t)_{t\geqslant 0}$ *be a family of invertible* $d \times d$-*matrices. Let us suppose that the following conditions hold (as* $t \to +\infty$):

(a) $\sum_{i,j}|a_t^{i,j}|\longrightarrow 0$.
(b) $\sum_i|a_t^{i,j}|E[\sup_{0\leqslant s\leqslant t}|\Delta_s M^j|]\longrightarrow 0$ *for each j.*
(c) $a_t Q_t a_t' \longrightarrow U$ *in probability (where U is a random variable with values in the space of positive semi-definite* $d \times d$-*matrices and* a_t' *denotes the transpose of* a_t).
(d) $a_t E[M_t M_t']a_t' \longrightarrow \Sigma$, *where* Σ *is a positive definite* $d \times d$-*matrix.*

Then the random vector $a_t M_t$ *converges* $\mathcal{A}$-*stably to the Gaussian kernel* $\mathcal{N}(0, U)$.

The concept of stable convergence was introduced by Rényi [14] and further developed by many authors: for instance, Aldous and Eagleson [1], Jacod and Memin [10].

In this paper, we present a version (see Theorem 2.2) of the central limit theorem for multivariate martingales which is more general than the one mentioned above. Indeed we eliminate some superfluous hypotheses. In particular, we suppress the assumption, which appears in all central limit theorems published so far, on the convergence of $a_t E[M_t M_t']a_t'$ (see condition (d) in Theorem 1.1). Our proof is based on the multidimensional version of a convergence result for triangular arrays proved by Letta and Pratelli [13].

We refer the interested reader to Crimaldi and Pratelli [3,4] for a stronger formulation of Theorem 2.2 in which, under the same assumptions of Theorem 2.2, we obtain the convergence of suitable conditional expectations.

Finally, it may be worthy of note to recall that the central limit theorem for multivariate martingales is a very useful tool in applications. It is used, for instance, in order to obtain asymptotic results in likelihood theory for stochastic processes (e.g. [2,7,11]).

2. Main theorem

Let us start with the following notation:

Notation 2.1. If a is a real matrix, we denote by $|a|$ the sum of the absolute values of its entries. With this notation, if a is equal to the row-column product of two matrices b, c, we have $|a| \leqslant |b|\,|c|$. Moreover, we denote by a' the transpose of a. A d-dimensional real vector x is identified with a column matrix: so the symbol $|x|$ denotes the sum of the absolute values of its entries. Hence, we have

$$|xx'| = |x|^2. \tag{1}$$

We can now state our main theorem.

Theorem 2.2. *On a probability space* $(\Omega, \mathscr{A}, P)$, *endowed with a filtration* $\mathscr{F} = (\mathscr{F}_t)_{t \geqslant 0}$ *which satisfies the usual conditions, let* $M = (M_t)_{t \geqslant 0}$ *be a (right-continuous with limits from the left) d-dimensional martingale. Let us denote by* Q *the process* $[M, M]$ *(with values in the space of positive semi-definite* $d \times d$*-matrices). Further let* $(a_t)_{t \geqslant 0}$ *be a family of* $d \times d$*-matrices. Let us suppose that the following conditions hold (as* $t \to +\infty$*):*

(a) $|a_t| \longrightarrow 0$.
(b) $E[\sup_{0 \leqslant s \leqslant t} |a_t \Delta_s M|] \longrightarrow 0$.
(c) $a_t Q_t a_t' \longrightarrow U$ *in probability (where* U *is a random variable with values in the space of positive semi-definite* $d \times d$*-matrices).*

Then the random vector $a_t M_t$ *converges* $\mathscr{A}$*-stably to the Gaussian kernel* $\mathscr{N}(0, U)$.

From Theorem 2.2 we immediately get the following corollaries:

Corollary 2.3. *With the same assumptions and notation as in Theorem* 2.2, *let us set* $A = \{\det U > 0\}$, $A_t = \{\det(a_t Q_t a_t') > 0\}$ *and* $B_t = \{\det Q_t > 0\}$. *Let us suppose* $P(A) > 0$ *and denote by* P_A *the probability measure* $P(\cdot|A)$. *Then, under* P_A, *we have the following* $\mathscr{A}$*-stable convergences*:

(a) $U^{-1/2} a_t M_t \longrightarrow \mathscr{N}(0, I_d)$.
(b) $I_{A_t} (a_t Q_t a_t')^{-1/2} a_t M_t \longrightarrow \mathscr{N}(0, I_d)$.
(c) $I_{B_t} M_t' Q_t^{-1} M_t \longrightarrow \chi^2(d)$.

Corollary 2.4. (Weak law of large numbers for martingales; cf. Corollary 2.2 in [12]).
With the same assumptions and notation as in the previous corollary, the random vector $I_{B_t} Q_t^{-1} M_t$ *converges in probability, under* P_A, *to zero.*

3. Preliminaries

In order to prove Theorem 2.2, we shall need the following "multidimensional version" of the result obtained in [13]. (It can be easily deduced using the Cramér–Wold device.)

Proposition 3.1. *On a probability space* $(\Omega, \mathscr{A}, P)$ *let* $(X_{n,j})_{n \geqslant 1,\, 1 \leqslant j \leqslant k_n}$ *be a triangular array of d-dimensional real random vectors, such that, for each* n, *the finite sequence* $(X_{n,j})_{1 \leqslant j \leqslant k_n}$ *is a martingale difference array with respect to a given filtration* $(\mathscr{G}_{n,j})_{j \geqslant 0}$.

Let us set

$$X_n^* = \sup_{1 \leqslant j \leqslant k_n} |X_{n,j}|, \qquad U_n = \sum_{j=1}^{k_n} X_{n,j} X'_{n,j}.$$

Moreover, let us denote by $\mathscr{H}_j$ *the algebra* $\liminf_n \mathscr{G}_{n,j}$ *and by* $\mathscr{U}$ *the* σ*-field generated by the algebra* $\bigcup_j \mathscr{H}_j$. *Let us suppose that* X_n^* *converges in* L^1 *to zero and that* U_n *converges in probability to a* $\mathscr{U}$*-measurable random variable* U (*with values in the space of positive semi-definite* $d \times d$*-matrices*).

Then $\sum_{j=1}^{k_n} X_{n,j}$ *converges* $\mathscr{U}$*-stably to the Gaussian kernel* $\mathscr{N}(0, U)$.

Further, the following lemmas will be useful.

Lemma 3.2. *Under the same assumptions as in Theorem* 2.2, *let us suppose* $M_0 = 0$ *and denote by* M^* *the real process defined by* $M_u^* = \sup_{0 \leqslant s \leqslant u} |M_s|$. *Further, for each positive number* t, *let us denote by* T_t *the first entrance time of the real process* $|a_t|^{1/2} M^* \vee |a_t|\,|Q|$ *into* $]1, \infty[$. *Finally, let us set*

$$S_t = T_t \wedge t. \tag{2}$$

Then the following statements hold:

(a) S_t *is a stopping time which is bounded by the constant* t.
(b) $S_t \longrightarrow +\infty$.
(c) $|a_t M_{S_t}|$ *converges in* L^1 *to zero.*
(d) $|a_t Q_{S_t} a'_t|$ *converges in probability to zero.*

Proof. Since the two processes M^*, $|Q|$ are adapted and right-continuous, T_t is a stopping time. So, because of (2), we get statement (a).

By assumption (a) in Theorem 2.2 and the fact that the two processes M^*, $|Q|$ have locally bounded trajectories, we obtain statement (b).

In order to prove statement (c), it suffices to observe that we have

$$M_{S_t} = M_{S_t-} + \Delta_{S_t} M, \qquad |a_t|^{1/2} |M_{S_t-}| \leqslant 1$$

and so

$$\begin{aligned} |a_t M_{S_t}| &\leqslant |a_t M_{S_t-}| + |a_t \Delta_{S_t} M| \\ &\leqslant |a_t|\,|M_{S_t-}| + \sup_{0 \leqslant s \leqslant t} |a_t \Delta_s M| \\ &\leqslant |a_t|^{1/2} + \sup_{0 \leqslant s \leqslant t} |a_t \Delta_s M|. \end{aligned}$$

Similarly, in order to prove statement (d), it suffices to observe that we have

$$Q_{S_t} = Q_{S_t-} + \Delta_{S_t} Q, \qquad |a_t|\,|Q_{S_t-}| \leqslant 1,$$

and so

$$\begin{aligned}|a_t Q_{S_t} a_t'| &\leqslant |a_t Q_{S_t-} a_t'| + |a_t(\Delta_{S_t} Q) a_t'| \\ &\leqslant |a_t|\,|Q_{S_t-}|\,|a_t| + \sup_{0\leqslant s\leqslant t} |a_t(\Delta_s Q) a_t'| \\ &\leqslant |a_t| + \sup_{0\leqslant s\leqslant t} |a_t(\Delta_s M)(\Delta_s M)' a_t'| \\ &= |a_t| + \sup_{0\leqslant s\leqslant t} |a_t \Delta_s M|^2\end{aligned}$$

(where the last equality follows from (1)). □

Lemma 3.3. *Let us adopt the same hypotheses and notation as in Lemma* 3.2. *Then, for each fixed positive number t and each strictly positive number* ε, *there exist an increasing sequence* $(T_j)_{j\geqslant 0}$ *of stopping times, with* $T_0 = S_t$, $S_t \leqslant T_j \leqslant t$, *and a strictly positive integer k, such that the array* $(X_j)_{1\leqslant j\leqslant k}$ *defined by*

$$X_j = a_t(M_{T_j} - M_{T_{j-1}}) \quad \textit{for } 1\leqslant j\leqslant k \tag{3}$$

(*which is a martingale difference array with respect to the filtration* $(\mathscr{F}_{T_j})_{j\geqslant 0}$) *has the following properties*:

$$P\left\{\left|a_t(M_t - M_{S_t}) - \sum_{j=1}^{k} X_j\right| > \varepsilon\right\} < \varepsilon, \tag{4}$$

$$P\left\{\left|a_t(Q_t - Q_{S_t})a_t' - \sum_{j=1}^{k} X_j X_j'\right| > \varepsilon\right\} < \varepsilon, \tag{5}$$

$$\sup_{1\leqslant j\leqslant k} |X_j| \leqslant \varepsilon + \sup_{0\leqslant s\leqslant t} |a_t \Delta_s M|. \tag{6}$$

Proof. Let us fix $t\geqslant 0$ and $\varepsilon > 0$. For each integer n, let us define (by induction) the increasing sequence $(S_{n,j})_{j\geqslant 0}$ of stopping times, with $S_t \leqslant S_{n,j} \leqslant t$, setting

$$S_{n,0} = S_t, \quad S_{n,j} = t \wedge (S_{n,j-1} + n^{-1}) \wedge U_{n,j},$$

where

$$U_{n,j}(\omega) = \inf\{s \in \mathbb{R} : s > S_{n,j-1}(\omega), |a_t(M_s(\omega) - M_{S_{n,j-1}}(\omega))| > \varepsilon\}. \tag{7}$$

Then it is a well-known fact that it is possible to find a pair (n, k) of strictly positive integers such that, setting $T_j = S_{n,j}$ and defining X_j by (3), conditions (4), (5) are satisfied. Moreover, condition (6) holds because of (7). □

Lemma 3.4. *On a measurable space* $(\Omega, \mathscr{A})$, *let* $\mathscr{F} = (\mathscr{F}_t)_{t\geqslant 0}$ *be a right-continuous filtration. Let us set* $\mathscr{F}_\infty = \bigvee_t \mathscr{F}_t$. *Further, let* $(T_n)_{n\geqslant 1}$ *be a sequence of finite stopping times with* $T_n \to +\infty$ *and let us denote by* $\mathscr{H}$ *the algebra* $\liminf_n \mathscr{F}_{T_n}$.

Then the σ-*field generated by* $\mathscr{H}$ *coincides with the whole* σ-*field* $\mathscr{F}_\infty$.

Proof. The sequence $(T_n)_{n\geqslant 1}$ is bounded from below by the increasing sequence

$$(T_n \wedge T_{n+1} \wedge \cdots)_{n\geqslant 1},$$

which goes to $+\infty$ and which is still a sequence of stopping times (because of the right-continuity of the filtration $\mathscr{F}$). Thus, without loss of generality, we may suppose that the sequence $(T_n)_{n\geqslant 1}$ itself is increasing. In this case, the statement of the lemma is well known. □

4. Proof of Theorem 2.2

We are now able to prove Theorem 2.2.

Without loss of generality, we may suppose $M_0 = 0$. Let us fix an increasing sequence $(t_n)_{n\geqslant 1}$ of real positive numbers with $t_n \uparrow +\infty$. It is enough to prove that $a_{t_n} M_{t_n}$ converges $\mathscr{F}_\infty$-stably to the Gaussian kernel $\mathscr{N}(0, U)$. To this end, we observe that, if we denote by S_n the stopping time S_{t_n} defined in Lemma 3.2, this lemma shows that S_n goes to $+\infty$ and the following two convergences hold in probability

$$|a_{t_n} M_{S_n}| \longrightarrow 0, \tag{8}$$

$$|a_{t_n} Q_{S_n} a'_{t_n}| \longrightarrow 0. \tag{9}$$

Therefore, it suffices to prove that $a_{t_n}(M_{t_n} - M_{S_n})$ converges $\mathscr{F}_\infty$-stably to the kernel $\mathscr{N}(0, U)$. To this end, let us apply Lemma 3.3 with t_n instead of t and with n^{-1} instead of ε. Thus, we obtain a double sequence $(T_{n,j})_{n\geqslant 1, j\geqslant 0}$ of stopping times and a sequence $(k_n)_{n\geqslant 1}$ of strictly positive integers such that, for each n, the sequence $(T_{n,j})_{j\geqslant 0}$ is an increasing sequence of stopping times with $T_{n,0} = S_n$, $S_n \leqslant T_{n,j} \leqslant t_n$ and, setting

$$X_{n,j} = a_{t_n}(M_{T_{n,j}} - M_{T_{n,j-1}}) \quad \text{for } 1 \leqslant j \leqslant k_n,$$

the following properties hold:

$$P\left\{ \left| a_{t_n}(M_{t_n} - M_{S_n}) - \sum_{j=1}^{k_n} X_{n,j} \right| > n^{-1} \right\} < n^{-1}, \tag{10}$$

$$P\left\{ \left| a_{t_n}(Q_{t_n} - Q_{S_n}) a'_{t_n} - \sum_{j=1}^{k_n} X_{n,j} X'_{n,j} \right| > n^{-1} \right\} < n^{-1}, \tag{11}$$

$$\sup_{1\leqslant j\leqslant k_n} |X_{n,j}| \leqslant n^{-1} + \sup_{0\leqslant s\leqslant t_n} |a_{t_n} \Delta_s M|. \tag{12}$$

Then, setting $\mathscr{G}_{n,j} = \mathscr{F}_{T_{n,j}}$, the triangular array $(X_{n,j})_{n\geqslant 1,\, 1\leqslant j\leqslant k_n}$, satisfies (with respect to $(\mathscr{G}_{n,j})_{n\geqslant 1, j\geqslant 0}$) the assumptions of Proposition 3.1. More precisely: from inequality (12) we deduce, by assumption (b) in Theorem 2.2, that the sequence

$$\left(\sup_{1\leqslant j\leqslant k_n} |X_{n,j}| \right)_{n\geqslant 1}$$

converges in L^1 to zero; from (9) and (11) we deduce, by assumption (c) in Theorem 2.2, that the sequence

$$\left(\sum_{j=1}^{k_n} X_{n,j} X'_{n,j}\right)_{n \geqslant 1}$$

converges in probability to U. Finally, since U is measurable with respect to $\mathcal{F}_\infty$, the condition of measurability required for U in Proposition 3.1 is obviously verified: indeed, for each j, the sequence $(T_{n,j})_{n\geqslant 1}$ (which is bounded from below by $(S_n)_{n\geqslant 1}$) goes to $+\infty$, and so, by Lemma 3.4, the σ-field generated by the algebra

$$\mathcal{H}_j = \liminf_n \mathcal{F}_{T_{n,j}}$$

(and, consequently, the σ-field $\mathcal{U}$) coincides with $\mathcal{F}_\infty$.

Thus, applying Proposition 3.1, we get that the sequence

$$\left(\sum_{j=1}^{k_n} X_{n,j}\right)_{n \geqslant 1} \tag{13}$$

converges $\mathcal{F}_\infty$-stably to the kernel $\mathcal{N}(0, U)$. Then, in order to conclude, it suffices to observe that thanks to (10), the sequence $(a_{t_n}(M_{t_n} - M_{S_n}))_{n\geqslant 1}$ differs from sequence (13) up to a sequence which converges in probability to zero.

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