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Stochastic Model Predictive Control for Constrained Networked Control Systems with Random Time Delay

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Abstract: In this paper the continuous time stochastic constrained optimal control problem is formulated for the class of networked control systems assuming that time delays follow a discrete-time, finite Markov chain . Polytopic overapproximations of the system's trajectories are employed to produce a polyhedral inner approximation of the non-convex constraint set resulting from imposing the constraints in continuous time. The problem is cast in a Markov jump linear systems (MJLS) framework and a stochastic MPC controller is calculated explicitly, offline, coupling dynamic programming with parametric piecewise quadratic (PWQ) optimization. The calculated control law leads to stochastic stability of the closed loop system, in the mean square sense and respects the state and input constraints in continuous time.

1. INTRODUCTION

A networked control system (NCS) is a feedback control system where its various components (sensor, actuator, controller, plant) are connected through a communication network. Due to their numerous advantages such as low installation and maintenance cost and high flexibility, they have attracted a lot of interest over the past few years (Antsaklis and Baillieul [2004], Baillieul and Antsaklis [2007], Heemels and van de Wouw [2010], Hespanha et al. [2007], Tipsuwan and Chow [2003], Zhang et al. [2001]). The presence of a communication network in the loop introduces many interesting phenomena such as random transmission delays and packet losses, which need to be taken into account by the control synthesis algorithm in order to guarantee closed-loop stability. For example the presence of time-varying transmission delays may render, an otherwise stable closed-loop system, unstable (Cloosterman [2008], Zhang et al. [2001]).

There are two approaches proposed in the literature concerning the modeling of the transmission delay. The one is to model it as a deterministic quantity that takes values in a closed interval (*deterministic approach*), (Cloosterman [2008], Cloosterman et al. [2006], Cloosterman et al. [2008], Cloosterman et al. [2009]). The second approach is to consider the delay as random (*stochastic approach*), (Antunes et al. [2009], Chen et al. [2008a], Chen et al. [2008b], Donkers et al. [2010], Montestruque and Antsaklis [2004], Nilsson [1998], Nilsson et al. [1998], Seiler and Sengupta [2005], Shi and Yu [2009], Zhang et al. [2005]). The first approach leads to an uncertain discrete-time system, with the uncertainty appearing in an exponential form. The next step of the deterministic approach is to construct a polytopic (possibly with an additive norm-bounded term) over-approximation of the exponential uncertainty, in order to arrive to a model that is amenable to robust stability analysis and control synthesis, using classical LMI (linear matrix inequalities) techniques. The stochastic approach uses a probabilistic description of the transmission delay, assuming that it is either an independent across time random variable that follows a continuous or discrete probability distribution, or a stochastic process such as a Markov chain. Therefore, stability analysis and control synthesis is performed in a stochastic setting with the appropriate notion of stability being that of mean-square stability.

On the other hand, constraints on the input and state of the system usually need to be imposed in any realistic setting. Usually constrained control problems are treated in a *model predictive control* (MPC) framework, by solving a finite-horizon constrained optimal control problem at each sampling instant. Assuming that the control action is imposed through zero-order hold (ZOH) to the system, incorporation of input constraints at the sampling instances to the MPC optimization problem is straightforward. However, since the actual system is continuous in time, the requirement that the state constraints must be satisfied at all times results in a non-convex optimization problem even for linear systems with polyhedral constraints. To the best of our knowledge, there are no practical approaches considered in the literature concerning constrained control of (even non-networked) sampled-data systems, where state constraints are imposed in the continuous time.

In this paper, we consider control of constrained, continuous-time linear systems through a communication network. We model the transmission delay from the sensor to controller as a Markov chain, while we assume that the delay from the controller to the actuator (including the controller delay itself) is constant by a buffering technique. Specifically, we assume that the controller has knowledge of the current sensor-to-controller delay at each time instant (as is usually the case via timestamping techniques), and that the value of the delay switches among a finite number of values according to a Markov chain. The system is subject to hard state and input constraints that must be fulfilled in continuous time by the closed-loop system. The model is cast in a Markovian switching framework, which is analyzed in the authors recent work, Patrinos and Sarimveis [2010]. The state constraints in the continuous time are approximated by imposing sufficient conditions that the state of the system must satisfy only at the sampling instances. The technique is based on recent ideas regarding polytopic approximations of systems with time-varying delays, (Cloosterman [2008], Gielen et al. [2009], Gielen et al. [2010], Heemels et al. [2010], Hetel et al. [2006], Hetel et al. [2007]).

Advantages of the proposed approach is that it guarantees fulfillment of constraints for the continuous-time system for all possible values of the transmission delay, without destroying the structure of the resulting optimal control problem, i.e. this is guaranteed by imposing the state to lie in a polyhedral set only at the sampling instances. Furthermore, the controller is allowed to switch synchronously with the sensor-to-controller delay providing larger flexibility and less conservativeness in comparison to the deterministic approach.

2. MODEL DEFINITION

2.1 NCS model

The NCS model consists of a linear, time-invariant, continuous-time plant and a discrete-time controller that are connected through a communication network with induced sensor-to-controller (SC), τ^{sc} and controller-to-actuator (CA), τ^{ca}, delays. The controller delay (the time needed by the controller to perform computations) is assumed to be incorporated into the CA delay. The full state of the system is sampled by a time-driven sensor with a constant sampling interval $h > 0$. The discrete-time controller is event-driven and able to monitor the SC delay, via timestamping. The CA delay is considered to be constant by using the buffering technique. The discrete-time control signal u_k is transformed to a continuous-time control input $u(t)$ by a zero-order hold device (ZOH). Based on these assumptions, the NCS model is:

$$\dot{x}(t) = A_{\mathrm{c}}x(t) + B_{\mathrm{c}}u(t) \tag{1a}$$

$$u(t) = u_k, \text{ for } t \in [kh + \tau_k^{\mathrm{sc}} + \tau_k^{\mathrm{ca}}, (k+1)h + \tau_{k+1}^{\mathrm{sc}} + \tau_{k+1}^{\mathrm{ca}}) \tag{1b}$$

with $A_{\mathrm{c}} \in \mathbb{R}^{n\times n}$, $B_{\mathrm{c}} \in \mathbb{R}^{n\times m}$. The SC delay evolves according to a discrete-time, time-homogeneous Markov chain $\{r_k\}_{k\in\mathbb{N}}$ taking values in a finite set $\mathcal{S} \triangleq \{1,\dots,S\}$ with transition matrix $P = (p_{ij}) \in \mathbb{R}^{S\times S}$. Therefore, τ_k^{sc} switches among a finite set of values, where the switching is orchestrated by the underlying Markov chain $\{r_k\}_{k\in\mathbb{N}}$. This assumption manages, in some sense, the correlation of the transmission delays across time that is present in many types of networks (Nilsson [1998], Tipsuwan and Chow [2003]). Since the CA delay is considered constant through the buffering technique, the two transmission delays can be lumped into $\tau_k \triangleq \tau_k^{\mathrm{sc}} + \tau_k^{\mathrm{ca}}$. Therefore, the evolution of the total transmission delay τ_k is governed by the underlying Markov chain $\{r_k\}_{k\in\mathbb{N}}$ and the controller has access to τ_k at time k, but only probabilistic information is available regarding its future evolution. Throughout the paper, it is assumed that $\tau_k \le h$. Furthermore, there is one-to-one correspondence between the possible values of τ_k and the states of the Markov chain, i.e.:

$$\tau_k = \tau^i \triangleq \tau^{\mathrm{sc},i} + \tau^{\mathrm{ca}}, \text{ if } r_k = i,\ i \in \mathcal{S} \tag{2}$$

The *cover* of a state $i \in \mathcal{S}$ is the set of all states accessible i in one time step, i.e. $\mathcal{S}_i \triangleq \{j \in \mathcal{S} | p_{ij} > 0\}$. An *admissible switching path* of length $N \in \mathbb{N}$, $\boldsymbol{r} \triangleq (r_0,\cdots,r_N)$ for $\{r_k\}_{k\in\mathbb{N}}$ is a switching path for which $r_{k+1} \in \mathcal{S}_{r_k}$, for any $k \in \mathbb{N}_{[0,N-1]}$. We denote by $\mathfrak{G}$ the set of all admissible switching paths (of infinite length), and by $\mathfrak{G}_N$ the set of all admissible switching paths of length N. For any $i \in \mathcal{S}$, $\mathfrak{G}(i) \triangleq \{\boldsymbol{r} \in \mathfrak{G} | r_0 = i\}$ and $\mathfrak{G}_N(i) \triangleq \{\boldsymbol{r} \in \mathfrak{G}_N | r_0 = i\}$ denote the set of all *admissible switching paths emanating from* i , of infinite length and length N, respectively. The state of the system generated by the NCS model (1) is required to belong to a polyhedral set X, i.e.

$$x(t) \in X,\ t \in \mathbb{R}_+ \tag{3}$$

Notice that constraints are imposed in the continuous time. Similarly, the input signal is required to belong to a polyhedral set U, i.e.

$$u_k \in U,\ k \in \mathbb{N} \tag{4}$$

Notice that since the input is a discrete-time signal, imposing the constraints only at the sampling instances is sufficient to guarantee satisfaction of input constraints in continuous time. We are interested in *mode-dependent policies* of the form $\pi = \{\mu_0, \mu_1, \dots\}$, with $\mu_k : \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}^m$. Given a policiy π and a switching delay path $\boldsymbol{r}$, the input signal in continuous-time is given by $u(t) = \mu_k(x(kh), r_k)$, for $t \in [kh + \tau_k, (k+1)h + \tau_{k+1})$. Given a policy π, an initial state $x(0) = x$, an initial delay $r_0 = i$, and an admissible switching path $\boldsymbol{r}$, the solution of (1) at time $t \in \mathbb{R}_+$ is denoted by $\psi(t; x, i, \pi, \boldsymbol{r})$.

2.2 Exact discrete-time system

The exact discretization of (1) is

$$x_{k+1} = \mathrm{e}^{A_{\mathrm{c}}h}\, x_k + \int_0^{h-\tau_k} \mathrm{e}^{A_{\mathrm{c}}s}\, ds B_{\mathrm{c}} u_k + \int_{h-\tau_k}^{h} \mathrm{e}^{A_{\mathrm{c}}s}\, ds B_{\mathrm{c}} u_{k-1}$$

where $x_k \triangleq x(kh)$ is the discrete-time state at the k_{th} sampling instant. The above system can be cast in standard state-space form with respect to the augmented state vector $\xi_k \triangleq [x_k'\ u_{k-1}']'$, (Cloosterman [2008], Nilsson [1998]).

$$\xi_{k+1} = A_{r_k}\xi_k + B_{r_k}u_k \tag{5}$$

with $A_i \triangleq \begin{bmatrix} \mathrm{e}^{A_{\mathrm{c}}h} & \int_{h-\tau^i}^{h} \mathrm{e}^{A_{\mathrm{c}}s}\, ds B_{\mathrm{c}} \\ 0 & 0 \end{bmatrix}$, $B_i \triangleq \begin{bmatrix} \int_0^{h-\tau^i} \mathrm{e}^{A_{\mathrm{c}}s}\, ds B_{\mathrm{c}} \\ I \end{bmatrix}$, $i \in \mathcal{S}$. Notice that (A_i, B_i), $i \in \mathcal{S}$ can be computed exactly. Therefore, the NCS model (1) has been transformed into the Markovian Jump Linear system (MJLS) (5) with respect to the augmented state vector ξ_k, via exact discretization.

2.3 Handling constraints in continuous time

Guaranteeing constraint fulfillment for the closed-loop system requires from the controller to take into consideration the so-called inter-sample behavior, i.e. what happens to the state vector between two sampling instants. Let $\tilde{t} \triangleq t-kh$. Assume that $r_k = i$. Then the inter-sample behavior is described by the following two equations (Cloosterman [2008]):

$$x(kh+\tilde{t}) = \Gamma_i^1(\tilde{t})\xi_k, \text{ for } \tilde{t} \in [0, \tau^i) \tag{6a}$$

$$x(kh+\tilde{t}) = \Gamma_i^2(\tilde{t}) \left[\begin{smallmatrix} \xi_k \\ u_k \end{smallmatrix} \right], \text{ for } \tilde{t} \in [\tau^i, h) \tag{6b}$$

where

$$\Gamma_i^1(\tilde{t}) \triangleq \left[\mathrm{e}^{A_c\tilde{t}} \ \int_0^{\tilde{t}} \mathrm{e}^{A_c s}\, ds B_c \right] \tag{7a}$$

$$\Gamma_i^2(\tilde{t}) \triangleq \left[\mathrm{e}^{A_c\tilde{t}} \ \int_{\tilde{t}-\tau^i}^{\tilde{t}} \mathrm{e}^{A_c s}\, ds B_c \ \int_0^{\tilde{t}-\tau^i} \mathrm{e}^{A_c s}\, ds B_c \right] \tag{7b}$$

Considering $\tilde{t}$ as an uncertain parameter appearing in the matrices $\Gamma_i^j(\tilde{t})$, $j = 1, 2$ in exponential form, one can compute an over-approximation separately for these two cases. Specifically, let $T_i^1 \triangleq [0, \tau^i)$, $T_i^2 \triangleq [\tau^i, h)$. Furthermore, let:

$$\Gamma_i^j \triangleq \{\Gamma_i^j(\tilde{t}) | \tilde{t} \in T_i^j\} \qquad , \ j = 1, 2 \tag{8a}$$

$$\tilde{\Gamma}_i^j \triangleq \left\{ \sum_{p=1}^{P_i^j} \alpha_p \Gamma_{i,p}^j \,\middle|\, \alpha \in \mathbb{R}_+^{P_i^j}, \sum_{p=1}^{P_i^j} \alpha_p = 1 \right\}, \ j = 1, 2 \tag{8b}$$

where $\Gamma_{i,p}^1 \in \mathbb{R}^{n\times(n+m)}$, $\Gamma_{i,p}^2 \in \mathbb{R}^{n\times(n+2m)}$ and P_i^j, $i \in \mathcal{S}$, $j = 1, 2$ are positive integers. The next lemma provides a sufficient condition imposed on the the state-input vector of the exact discretization (5) such that the state constraints (3) of the NCS (1) are satisfied in continuous time. Its proof is omitted due to lack of space.

Lemma 1. Let $r_k = i$ and Γ_i^j, $\tilde{\Gamma}_i^j$ be as in (8) with $\Gamma_i^j \subseteq \tilde{\Gamma}_i^j$, $j = 1, 2$. Let $\tilde{X}_i \triangleq (X_i^1 \times \mathbb{R}^m) \cap X_i^2$, where $X_i^j \triangleq \bigcap_{p\in\mathcal{P}_i^j} (\Gamma_{i,p}^j)^{-1}(X)$, $j = 1, 2$. Then:

$$(\xi_k, u_k) \in \tilde{X}_i \Rightarrow x(t) \in X, \ \forall\, t \in [kh, (k+1)h) \tag{9}$$

Notice that $\tilde{X}_i$ is a polyhedral set. For every $i \in \mathcal{S}$, let:

$$Y_i \triangleq \tilde{X}_i \cup (\mathbb{R}^n \times U) \tag{10}$$

Then, it follows from lemma 1 that the state and input constraints (3), (4) for the continuous time system are satisfied for all possible delay switching paths if:

$$(\xi_k, u_k) \in Y_{r_k}, \ k \in \mathbb{N} \ \forall\, \boldsymbol{r} \in \mathfrak{G} \tag{11}$$

Summing up, the polyhedral state constraint set of the continuous-time NCS (1), is inner-approximated with a family of mode-dependent polyhedral sets that capture the dependence on the random time delay τ_k , with respect to the state vector and input vector of the exact discretization (5).

2.4 Polytopic Overapproximation of the trajectories of an NCS

In this section we will present a method for the overapproximation of Γ_i^j , by polytopes of matrices, $\tilde{\Gamma}_i^j$. Taking into account the special structure of $\Gamma_i^j(t)$, this boils down to overapproximating functions of the general form $\Delta_0(t) \triangleq \mathrm{e}^{A_c t}$ for $t \in [t_1, t_2]$ and $\Gamma_0(t) \triangleq \int_0^t \mathrm{e}\, A_c s ds B_c$, i.e., we have to determine sets C_{Δ_0} and C_{Γ_0} such that for all $t \in [t_1, t_2]$, $\{\Delta_0(t) | t \in [t_1, t_2]\} \subseteq C_{\Delta_0}$ and $\{\Gamma_0(t) | t \in [t_1, t_2]\} \subseteq C_{\Gamma_0}$.

Heemels et al. [2010] provide a thorough literature overview on that matter from which the approach based on the canonical Jordan form decomposition and the Cayley-Hamilton Theorem provide a framework for the calculation of such polytopic overapproximations. Other approaches that employ truncated Taylor series expansions lead to non-polytopic norm bounded overapproximations. Finally, the method of Gridding and Bounding (G&B) Heemels et al. [2010] yields tight overapproximations and can be employed to obtain arbitrarily tight ϵ-overapproximations by partitioning the interval $[t_1, t_2]$ using a sequence of points $t_1 = \tau_0 < \tau_1 < \cdots < \tau_s = t_2$ and applying the overapproximation procedure on each subinterval $[t_l, t_{l+1}]$, $l \in \mathbb{N}_{[0,s-1]}$.

Applying Jordan decomposition on A_c , we obtain the equivalent representation $A_c = QJQ^{-1}$, hence $\Delta_0(t)$ and $\Gamma_0(t)$ are rewritten as $\Delta_0(t) = Q\,\mathrm{e}^{Jt} Q^{-1}$, $\Gamma_0(t) = Q \int_0^t \mathrm{e}^{Jz}\, dz Q^{-1} B$. The special structure of the Jordan matrix J allows for the analytical calculation of e^{Jt} and its integral $\int_0^t \mathrm{e}^{Jz}\, dz$ and via simple algebraic manipulations upper and lower bounds can be calculated element-wise. In particular, $\Delta_0(t)$ and $\Gamma_0(t)$ are decomposed to $\Delta_0(t) = \sum_{i=0}^{\kappa} \varphi_i(t) S_i$ and $\Gamma_0(t) = \sum_{i=0}^{\kappa} \gamma_i(t) S_i B$, where S_i are sparse matrices with $0-1$ entries. Exploiting the analytical formulas of $\varphi_i(t)$ and $\gamma_i(t)$ we calculate upper and lower bounds for them on $[t_1, t_2]$, that is:

$$\underline{\gamma}_i \le \gamma_i(t) \le \overline{\gamma}_i \text{ and } \underline{\varphi}_i \le \varphi_i(t) \le \overline{\varphi}_i \tag{12}$$

This leads to the following overapproximations:

$$\{\Delta_0(t) | t \in [t_1, t_2]\} \subseteq C_{\Delta_0} \triangleq \mathrm{conv}\{F_j\}_{j\in\mathbb{N}_{[1,2^\kappa]}} = \left\{ \textstyle\sum_{i=1}^{\kappa} \mu_i S_i | \mu_i \in \{\underline{\varphi}_i, \overline{\varphi}\} \right\}$$

$$\{\Gamma_0(t) | t \in [t_1, t_2]\} \subseteq C_{\Gamma_0} \triangleq \mathrm{conv}\{G_j\}_{j\in\mathbb{N}_{[1,2^\kappa]}} = \left\{ \textstyle\sum_{i=1}^{\kappa} \mu_i S_i B | \mu_i \in \{\underline{\gamma}_i, \overline{\gamma}\} \right\}$$

However, this approach introduces a constant overapproximation error. To surmount this drawback, we partition equidistantly the interval $[t_1, t_2]$ into s subintervals $T_i \triangleq [\tau_l, \tau_{l+1}]$ thus obtaining sets $C_{\Delta_0,l}$ and $C_{\Gamma_0,l}, l \in \mathbb{N}_{[0,s-1]}$. Then the overall overapproximations are $C_{\Gamma_0}^s \triangleq \mathrm{conv}\{C_{\Gamma_0,l}\}_{l\in\mathbb{N}_{[0,s-1]}}$ and $C_{\Delta_0}^s \triangleq \mathrm{conv}\{C_{\Delta_0,l}\}_{l\in\mathbb{N}_{[0,s-1]}}$.

3. FROM CONTINUOUS-TIME TO DISCRETE-TIME STOCHASTIC OPTIMAL CONTROL

In this section we formulate the constrained optimal control problem for the NCS as a stochastic control problem in continuous time. However, since the input to the system is a signal in discrete time, the expected outcome from the solution to the optimization problem is an optimal control policy $\pi^\star \triangleq \{\mu_0^\star, \mu_1^\star, \ldots\}$. Given an initial state $x(0) = x$, an initial mode $r_0 = i$ and a policy $\pi \triangleq \{\mu_0, \mu_1, \ldots\}$, we introduce the infinite horizon cost function:

$$V_\pi(x, i) \triangleq \mathbb{E}\left[\int_0^\infty g(x(t), u(t)) dt \right] \tag{13}$$

where g is a convex quadratic function of u, i.e., $g(x, u) \triangleq \frac{1}{2}(x'Qx + u'Ru)$ (Q and R are positive semidefinite and positive definite respectively), $x(t) \triangleq \psi(t;x,i,\pi,\boldsymbol{r})$ and $u(t) \triangleq \mu_k(x_k,r_k)$ for $t \in [kh+\tau_k,(k+1)h+\tau_{k+1})$ (piecewise constant using a ZOH element). The constrained infinite horizon stochastic control problem for the NCS is then formulated as follows:

$$V^\star(x,i) \triangleq \inf\{V_\pi(x,i)|\pi \in \Pi(x,i)\} \tag{14}$$

where $Pi(x,i)$ is the set of admissible policies for initial state x and initial mode i, i.e.:

$$\Pi(x,i) \triangleq \left\{\pi \;\middle|\; \begin{array}{c} \psi(t;x,i,\pi,\boldsymbol{r}) \in X, t \in \mathbb{R}_+ \\ \mu_k(\psi(t;x,i,\pi,\boldsymbol{r}),r_k) \in U, k \in \mathbb{N}, \forall \boldsymbol{r} \in \mathfrak{G}(i) \end{array}\right\}$$

The continuous-time stochastic optimal control problem (14) can be transformed into a discrete-time one exploiting the exact discretization of the system. The infinite horizon cost function (13) is restated as:

$$V_\pi(x,i) = \mathbb{E}\left[\sum_{k=0}^{\infty} \tfrac{1}{2}\int_{kh}^{(k+1)h} (x(t)'Qx(t) + u(t)'Ru(t))dt\right]$$

Some simple algrebraic manipulations yield:

$$V_\pi(\xi,i) = \mathbb{E}\left[\sum_{k=0}^{\infty} \ell(\xi_k,u_k,r_k)\right] \tag{15}$$

where

$$\ell(\xi,u,i) \triangleq \tfrac{1}{2}\left[\begin{smallmatrix}\xi\\u\end{smallmatrix}\right]' \left[\begin{smallmatrix}Q_i & S_i\\ S_i' & R_i\end{smallmatrix}\right] \left[\begin{smallmatrix}\xi\\u\end{smallmatrix}\right] \tag{16}$$

The formulas for the matrices appearing in (16) are omitted due to lack of space.

4. FINITE-HORIZON CONSTRAINED STOCHASTIC OPTIMAL CONTROL FOR NCS

The continuous-time NCS (1) subject to constraints (3) and (4) has been transformed to the Markovian switching system (5) subject to the mode dependent polyhedral constraint sets (11). Notice that the discrete-time systemẽqrefeq:MJLS has resulted from exact discretization, and approximation is performed with respect to the continuous-time state constraints only. Therefore, the objective has now become to design an optimal controller (in some sense) for the MJLS (5). This problem is a special case of the generic framework introduced in Patrinos and Sarimveis [2010]. Specifically, since individual mode dynamics are linear and the mode-dependent constraint sets are polyhedral, the MPC problem can be solved explicitly by coupling dynamic programming with the convex parametric piecewise quadratic solver of Patrinos and Sarimveis [2011].

For each $i \in \mathcal{S}$, let $\mathcal{U}_i(\xi) \triangleq \{u \in \mathbb{R}^m|(\xi,u) \in Y_i\}$ and $\Xi_i \triangleq \mathrm{dom}\,\mathcal{U}_i$. Let $Y \triangleq \{Y_i\}_{i\in\mathcal{S}}$ and $\Xi \triangleq \{\Xi_i\}_{i\in\mathcal{S}}$. A mapping $\mu : \mathbb{R}^{n+m} \times \mathcal{S} \to \mathbb{R}^m$, such that $\mu(\xi,i) \in \mathcal{U}_i(\xi)$ for each $\xi \in \Xi_i$, $i \in \mathcal{S}$, is called a *(mode-dependent) control law*. An infinite sequence of control laws $\pi = \{\mu_0,\mu_1,\ldots\}$ is called a *policy*. If the policy is of the form $\{\mu,\mu,\ldots\}$ then it is called *stationary* and is denoted by μ. Since we are only dealing with mode-dependent control laws and policies, the adjective mode-dependent will be omitted for brevity for the rest of the paper.

The solution of (5) at time k given a policy π and a switching path $\boldsymbol{r}$ with $r_0 = i$ and $\xi_0 = \xi$ is denoted by $\phi(k;\xi,i,\pi,\boldsymbol{r})$. For $\xi \in \Xi_i$, $i \in \mathcal{S}$, we denote by $\Pi_N(\xi,i)$ the set of admissible policies of length N, i.e.

$$\Pi_N(\xi,i) \triangleq \left\{\pi \;\middle|\; \begin{array}{r} \xi_k = \phi(k;\xi,i,\pi,\boldsymbol{r}), k \in \mathbb{N}_{[0,N]} \\ (\xi_k,\mu_k(\xi_k,r_k)) \in Y_{r_k}, k \in \mathbb{N}_{[0,N-1]} \\ \xi_N \in \Xi^f_{r_N} \forall \boldsymbol{r} \in \mathfrak{G}_N(i) \end{array}\right\}$$

Here $\Xi^f \triangleq \{\Xi^f_i\}_{i\in\mathcal{S}}$ is the mode-dependent terminal set, to which the state of the system needs to be steered at the end of the horizon. The finite horizon cost of the admissible policy $\pi \in \Pi_N(\xi,i)$ for (5) is:

$$V_{N,\pi}(\xi,i) \triangleq \mathbb{E}\left[\sum_{k=0}^{N-1} \ell(\xi_k,u_k,r_k) + V_f(\xi_N,r_N)\right] \tag{17}$$

where $\xi_k \triangleq \phi(k;\xi,i,\pi,\boldsymbol{r})$, $u_k \triangleq \mu_k(\xi_k,r_k)$ and the terminal cost $V_f(\xi,i) \triangleq \xi' P^f_i \xi$ with P^f_i positive semidefinite, $i \in \mathcal{S}$. The constrained finite horizon stochastic optimal control problem is:

$$\mathbb{P}_N(\xi,i): \quad V^\star_N(\xi,i) \triangleq \inf_{\pi\in\Pi_N(\xi,i)} V_{N,\pi}(\xi,i) \tag{18}$$

We call $V^\star_N$ the finite horizon value function. Notice that optimization is sought over truly closed-loop policies. Taking into account lemma 1, this guarantees satisfaction of the constraints for the continuous-time system for every admissible transmission delay path. The multi-stage problem $\mathbb{P}_N(\xi,i)$ can be decomposed in one-stage problems and solved using dynamic programming (DP). The dynamic programming algorithm for (18) is:

$$V^\star_0 = V_f \tag{19a}$$

$$V^\star_{k+1}(\xi,i) = \inf_u\{\ell(\xi,u,i) + \sum_{j\in\mathcal{S}} p_{ij}V^\star_k(A_i\xi + B_iu,j)\},$$
$$i \in \mathcal{S},\ k \in \mathbb{N}_{[0,N-1]} \tag{19b}$$

Since the individual mode dynamics of the Markovian switching system (5) are linear, the mode-dependent constraints (11) are polyhedral, and the stage cost (16) and terminal cost V_f are convex quadratic, it follows from theorem VI.I of Patrinos and Sarimveis [2010], that $V^\star_k(\cdot,i)$ are convex PWQ for each $i \in \mathcal{S}$ and each $k \in \mathbb{N}_{[1,N]}$. Furthermore, the DP subproblems can be explicitly solved off-line by the convex parametric PWQ optimization algorithm of Patrinos and Sarimveis [2011]. The algorithm decomposes the state space into a finite number of critical regions in a graph traversal framework, using graphical derivative (Rockafellar and Wets [2009]) formulas of solution mappings. For each $k \in \mathbb{N}_{[0,N-1]}$, the proposed algorithm calculates the optimal mode-dependent control law $\mu^\star_k$ as a piecewise affine (PWA) mapping for each mode $i \in \mathcal{S}$, i.e. $\mathrm{dom}\,V^\star_k(\cdot,i)$ is decomposed in a finite number of polyhedral sets $\{\mathcal{R}^j_{k,i}\}_{j\in\mathscr{J}_{k,i}}$ ($\mathscr{J}_{k,i}$ is a finite index set) on each of which $\mu^\star_k$ is affine, i.e. $\mu^\star_k(\xi,i) = K^j_{k,i}\xi + \kappa^j_{k,i}$, if $\xi \in \mathcal{R}^j_{k,i}$.

5. STOCHASTIC MPC OF CONSTRAINED NCS

Since $\mathbb{P}_N(\xi,i)$ is solved explicitly using dynamic programming and the convex parametric PWQ solver, at the last step of the algorithm we obtain the explicit stochastic MPC (*SMPC*) law, i.e. the control law that would result by solving repetitively online the problem $\mathbb{P}_N(\xi_k,r_k)$ at each time instant k, and applying to the system $\mu^\star_N(\xi_k,r_k)$. In fact the procedure could be performed on-line by reformulating $\mathbb{P}_N(\xi_k,r_k)$ as a multi-stage stochastic optimization problem (see section VI.B of Patrinos and Sarimveis

[2010]) using a scenario-based approach. The resulting optimization problem is a large-scale quadratic program, with an exponential increase in complexity with respect to the prediction horizon. Instead, the explicit DP approach decomposes the multi-stage problem into smaller single-stage problems, and performs all the heavy computations off-line. Therefore, online, only the evaluation of a PWA affine mapping is needed in order to compute the MPC law. For future reference, the following notation for the Markovian switching system in closed-loop with the SMPC controller is introduced:

$$\xi_{k+1} = A_{r_k}\xi_k + B_{r_k}\mu_N^\star(\xi_k, r_k) \tag{20}$$

6. STABILITY AND INVARIANCE PROPERTIES OF STOCHASTIC MPC FOR NCS

In this section we provide conditions that the terminal cost and terminal set must satisfy in order for the closed-loop system (20) with SMPC to be *mean-square (MS) stable*. Paralleling the theory of deterministic MPC for linear systems, the terminal weighting matrices P_i^f can be selected as the solution of the infinite horizon unconstrained stochastic optimal control problem, i.e. as the solution of the CARE Costa et al. [2005], ch.4):

$$P_i^f = A_i'\mathcal{E}_i(P^f)Ai - (A_i'\mathcal{E}_i(P^f)B_i + S_i)\cdot \tag{21}$$
$$(R_i + B_i'\mathcal{E}_i(P^f)B_i)^{-1}(B_i'\mathcal{E}_i(P^f)A_i + S_i) + Q_i,\ i \in \mathcal{S}$$

where $\mathcal{E}_i(P^f) \triangleq \sum_{j\in S} p_{ij}P_i^f$, and $\Xi^f = \{\Xi_i^f\}_{i\in\mathcal{S}}$ is the maximal uniformly positive invariant set for the Markovian switching system in closed loop with the unconstrained optimal policy $\mu(\xi, i) \triangleq -(R_i + B_i'\mathcal{E}_i(P^f)B_i)^{-1}(B_i'\mathcal{E}_i(P^f)A_i + S_i)\xi$ (Patrinos and Sarimveis [2010]). The following theorem provides the promised stabilizing properties of the SMPC controller. Its proof can be found in Patrinos and Sarimveis [2010].

Theorem 2. Suppose that the terminal cost and terminal set are selected as above for the Markovian switching system (5) subject to (11). Then the origin is MS stable in $\Xi_N^\star \triangleq domV_N^\star$ for the system in closed loop with the SMPC controller (cf. (20)). Furthermore, if $0 \in \operatorname{int}\Xi_i^f$, $i \in \mathcal{S}$ then the origin is exponentially MS stable in $\Xi_N^\star$.

Theorem 2 in turn implies MS stability of the origin for the continuous time NCS (1) in closed-loop with the SMPC controller, since using lemma 4.3.5 of Cloosterman [2008], that provides an upper bound for the norm of the state of the continuous time system between two sampling instants of the form $||x(kh + r)|| \le c_0||x_k|| + c_1||u_k|| + c_2||u_{k-1}||$, one concludes that $\lim_{t\to\infty} \mathbb{E}[x(t)]^2 = 0$.

7. EXAMPLE

In this section we apply the proposed method on an applied control problem and manifest its advantages over alternative approaches found in literature (Cloosterman [2008]). The NCS describes a printer that is controlled through a network. The matrices in (1) are $A_c = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $B_c = \left[\begin{smallmatrix} 0 \\ 126.70 \end{smallmatrix}\right]$, while $X = [-10, 10]^2$, $U = [-2, 2]$, $Q = 10I_2$ and $R = 1$. The sampling interval is $h = 20$ ms while the SC delay can take the values $\tau^{\text{sc},1} = 3$ ms and $\tau^{\text{sc},2} = 15$ ms with transition matrix $P = \left[\begin{smallmatrix} 0.67 & 0.33 \\ 0.30 & 0.70 \end{smallmatrix}\right]$. The CA delay is considered constant with $\tau^{\text{ca}} = 1$ ms. We set the prediction horizon to $N = 10$ steps. In the following illustrations we present a visualization of the polyhedral decomposition of the feasible state space on which the control law is defined as a PWA function over these regions. The mode-dependent PWA control law consists of 61 and 73 critical regions (cf. Figure 1) for each of the two modes.

In order to elucidate the benefits of SMPC we compare our results with alternative control approaches. The first approach (*Delay-free MPC*) is a deterministic MPC scheme for the exact discretization of the continuous-time system without taking into consideration the time-varying delay i.e., for the system $x_{k+1} = e^{A_c h}x_k + \Gamma_0(h)$. Constraints are imposed only on discrete sampling times while the cost function is considered to be quadratic, $\ell(x, u) = \frac{1}{2}(x'Q_h x + u'R_h u)$ where $Q_h = hQ$ and $R_h = hR$. The second alternative scheme (*Non-switched MPC*) is a deterministic MPC controller for the exact discretization of the continuous-time system where the delay is considered constant and equal to its greatest value (worst case scenario, $\tau_{\max} = 16$ ms), i.e. for the discrete-time system $\xi_{k+1} = \left[\begin{smallmatrix} e^{A_c h} & \Gamma_0(h) - \Gamma_0(h-\tau_{\max}) \end{smallmatrix}\right]\xi_k + \Gamma_0(h - \tau_{\max})u_k$ and the constraints are imposed only for the sampling times.

In order to compare SMPC against the alternative schemes, 20 simulations (corresponding to 20 switching paths according to the transition matrix) for every extreme point of the effective domain of $V_N^\star(\cdot, i)$, $i \in \mathcal{S}$ are performed. For every single one of them, SMPC achieved stochastic stability in the mean-square sense for the continuous time closed loop system while respecting the constraints in the continuous time. Non-switched MPC achieved this goal only in 66.77% of the cases while for delay-free MPC the percentage drops to 8.47%. An illustrative simulation of the NCS in closed-loop with the SMPC controller is depicted in Figure 2.

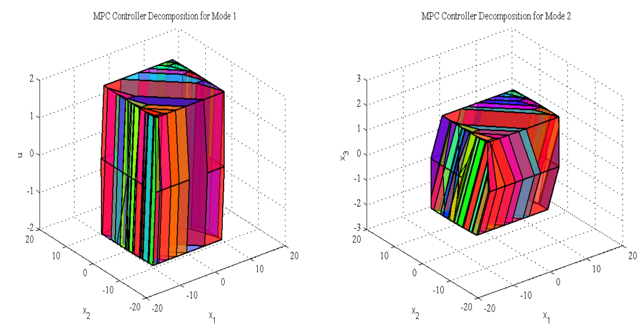

Fig. 1. PWA control law over a polyhedral decomposition of the extended state space for the network controlled printer.

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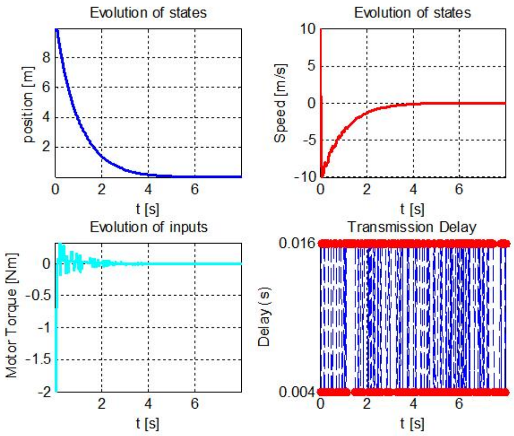

Fig. 2. Simulation of the closed-loop system using the SMPC controller, in continuous time, starting from $x(0) = [9.72\ 8.98]'$ and $r_0 = 1$.

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