Pathways towards instability in financial networks

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There is growing consensus that processes of market integration and risk diversification may come at the price of more systemic risk. Indeed, financial institutions are interconnected in a network of contracts where distress can either be amplified or dampened. However, a mathematical understanding of instability in relation to the network topology is still lacking. In a model financial network, we show that the origin of instability resides in the presence of specific types of cyclical structures, regardless of many of the details of the distress propagation mechanism. In particular, we show the existence of trajectories in the space of graphs along which a complex network turns from stable to unstable, although at each point along the trajectory its nodes satisfy constraints that would apparently make them individually stable. In the financial context, our findings have important implications for policies aimed at increasing financial stability. We illustrate the propositions on a sample dataset for the top 50 EU listed banks between 2008 and 2013. More in general, our results shed light on previous findings on the instability of model ecosystems and are relevant for a broad class of dynamical processes on complex networks.

Keywords: Financial networks, Systemic risk, Stability analisys, Policy making

The systemic risk emerging from networks of financial institutions (in the following *banks* for brevity) poses significant scientific challenges and comes with prominent policy and societal implications [5]. Banks are interconnected among each other both through direct exposures to bilateral contracts [6, 7] and through indirect exposures to common assets [8, 9]. Here we focus on direct interbank exposures, such as loans extended from one bank to another one, which are usually modeled as directed weighted networks [10–12], while the analysis could be later extended to indirect common exposures [9, 13].

While many factors drive systemic risk, the literature has identified two main channels for the propagation of financial distress through direct exposures. The first is known as illiquidity contagion: If banks anticipate that their counterparties may incur losses, they will try to withdraw their liquid funds from them [14], inducing them, in turn, to withdraw their funds from their own counterparties. Therefore, distress propagates from lenders to borrowers as their liquidity decreases. The second channel is the deterioration of interbank assets: lenders reassess the value of their claims towards borrowers under distress, taking into account the possibility that borrowers will not be able to meet their obligations. Devaluation of assets effectively generates losses for lenders, which can in turn be transmitted to their creditors [6, 15–17]. The propagation of distress is typically mitigated by the intervention of central banks, so that cascades of defaults are rare [18]. Yet, the process of illiquidity contagion is essentially driven by the anticipation of the *potential* interbank asset deterioration. Therefore, we focus on the latter mechanism only, in line with most of the previous literature [6, 11, 12].

A growing body of work [17, 19–24] carries out stress tests on the financial system by computing the distribution of losses conditional upon a given pattern of shocks. However, such an approach relies on specific assumptions on the nature of the financial contracts and the distress propagation mechanism. In contrast, here we derive, under mild assumptions, a very general and powerful result on the relation between network structure and stability of the system, and we show how pathways towards instability may emerge in the process of risk diversification. The findings yield specific policy insights that hold valid regardless of the details of financial contracts. We emphasize that, even though here we focus on the interbank network, our results are based on the study of the linear stability around a fixed point and therefore apply to a large class of dynamical processes on networks.

INTERBANK NETWORK

The *equity* E of a bank, i.e. the difference between its total assets and liabilities, is an important variable in determining the financial health of a bank. In the literature on financial contagion [6, 16, 17], a bank defaults as soon as its equity becomes negative, as it is unlikely that it will be able to repay its debts in full. The ratio between total assets and equity is called *leverage* and it is a coarse estimate of the riskiness of a bank, as it is related to the maximum loss on the assets that can be absorbed by the equity of the bank. While leverage is usually understood as a single number for each bank, the notion has been recently extended into the concept of *leverage matrix* [23], whereby leverage is computed with respect to each specific asset class or counterparty. In particular, for a system of n banks here we consider the $n \times n$ *interbank leverage matrix* Λ, whose elements Λ_{ij} are equal to the ratio between the nominal exposure of bank i towards bank j and the equity of bank i. The total interbank leverage of bank i is simply equal to $\ell_i = \sum_j \Lambda_{ij}$. In fact, we will consider an adjusted interbank leverage matrix $\hat{\Lambda}_{ij} = \Lambda_{ij}\,(1-\rho_j)$, where ρ_j is the recovery rate of bank j, i.e. the fraction of its interbank assets recovered by creditors in case of default. Finally, let us denote the relative equity losses of bank i as $h_i(t) = (E_i(0) - E_i(t))/E_i(0)$.

Starting from basic principles of financial accounting and under mild assumptions on the type of financial contracts among banks, we show that the relative equity loss of bank i must be a function of the relative equity loss of its counterparties, as well as function of the leverage matrix Λ_{ij}, according to the following dynamics: $h_i(t+1) = h_i(1) + \sum_j \hat{\Lambda}_{ij} p(h_j(t))$, where p is the default probability of counterparty j as a function of its relative equity loss (see SI Appendix for the details). Even though the function p is, in general, complicated and nonlinear, the response of the system to a small initial shock can be characterized by linear stability analysis around a fixed point. This implies that the system is unstable if and only if $\lambda_{\max}$, the largest eigenvalue of the Jacobian of the dynamics $\tilde{\Lambda}_{ij} = \hat{\Lambda}_{ij} p'_j(h_j(0))$, is larger than one. In the following, without loss of generality, we assume $p'_j(h_j(0)) = 1$, $\rho_j = 0$ for all j, and we denote $\tilde{\Lambda} = \Lambda$, although all the results hold in the more general case. Notice that the presence of a recovery rate pushes the system towards stability, as the largest eigenvalue of Λ is larger than or equal to the largest eigenvalue of $\hat{\Lambda}$. Notice also that under the further assumption that $p(h) = h$, we would recover the extension of the DebtRank dynamics [17, 23, 24] that includes a recovery rate (see SI Appendix).

Despite the considerable body of work on financial contagion, since there is no simple relationship between the topology of a network and $\lambda_{\max}$, the study of stability has been seldom carried out [25] in this context. Here, we show two important effects pertaining financial instability that had remained uncovered so far and could have profound policy implications. First, even if the individual leverage of banks does not increase, a financial system can turn from stable to unstable as the number of banks increases (i.e. the number of nodes in the network grows larger) like during a process of market integration. Second, even if the individual leverage of banks does not increase, a financial system can become unstable as the number of contracts among banks increases (i.e. the number of edges in the network increases) like during a process of risk diversification. Notably, in both cases instability appears despite the fact that the assessment that each bank makes of its own risk profile does not change, because individual leverage levels remain constant. This means that market integration and risk diversification can make the system *as a whole* unstable. These results do not imply that such processes are detrimental *per se*, but that financial policies focusing only on individual banks, also known as micro-prudential policies, can have the opposite effect of increasing financial instability if they do not consider the system as a whole. As it will be clear further below, the origin of instability lies in the fact that in both processes banks get increasingly involved in multiple cycles (i.e. closed chains) of contracts. Our results suggest to include the eigenvalue analysis of the leverage matrix among the tools of financial stability.

EMERGENCE OF INSTABILITY

The relation between $\lambda_{\max}$ and interbank leverage across banks becomes simple if all banks have the same interbank leverage or if the interbank network is a large Erdős-Rényi graph [27]. In the first case, via the Perron-Frobenius theorem, $\lambda_{\max}$ is bounded by the smallest and largest sum over the columns of the interbank leverage matrix, i.e. precisely by the smallest and largest interbank leverages. Hence, if all banks have the same interbank leverage ℓ, it must be also equal to $\lambda_{\max}$. The second case is similar to the May-Wigner theorem about the instability of model ecosystems [28] in which species interact through a large Erdős-Rényi graph. The main difference is that in our case interactions between banks are described by the leverage matrix Λ, which is non-negative, while the interactions between species in ecosystems are described by a matrix whose elements can have unspecified sign. In the SI Appendix we prove that, for $n \to \infty$, in this case $\lambda_{\max} \to \ell = \sum_i \ell_i/n = \sum_{i,j} \Lambda_{ij}/n$, the average interbank leverage across banks. Therefore, in both cases the system is unstable whenever $\ell > 1$.

When relaxing either of the two assumptions (homogeneity of leverage, or large size together with randomness of the graph), finer details of the network structure become important. For instance, because the theorem only holds in the limit of large size graphs, there exist small Erdős-Renyi graphs that are stable although they have $\ell > 1$. An example of a small size network that is extremely important for policy is the the network of the Global Systemically Important Banks [29], comprising about 30 banks. Suppose to start from a small and stable Erdős-Renyi graph with $\ell > 1$ and to connect more banks to the network (by keeping ℓ and the number of contracts per bank constant). Eventually, the system will grow large enough to become unstable because the theorem will have to hold in the limit of large graphs (see SI Appendix for an example). This is an example of a previously unreported phenomenon that we call *pathways to instability*, i.e. the existence of trajectories in the space of graphs along which financial networks turn from stable to unstable, although at each point along the trajectory banks individually satisfy constraints that would apparently make them individually stable.

In general, the system is unstable if and only if there exists an unstable *strongly connected component* (i.e. a

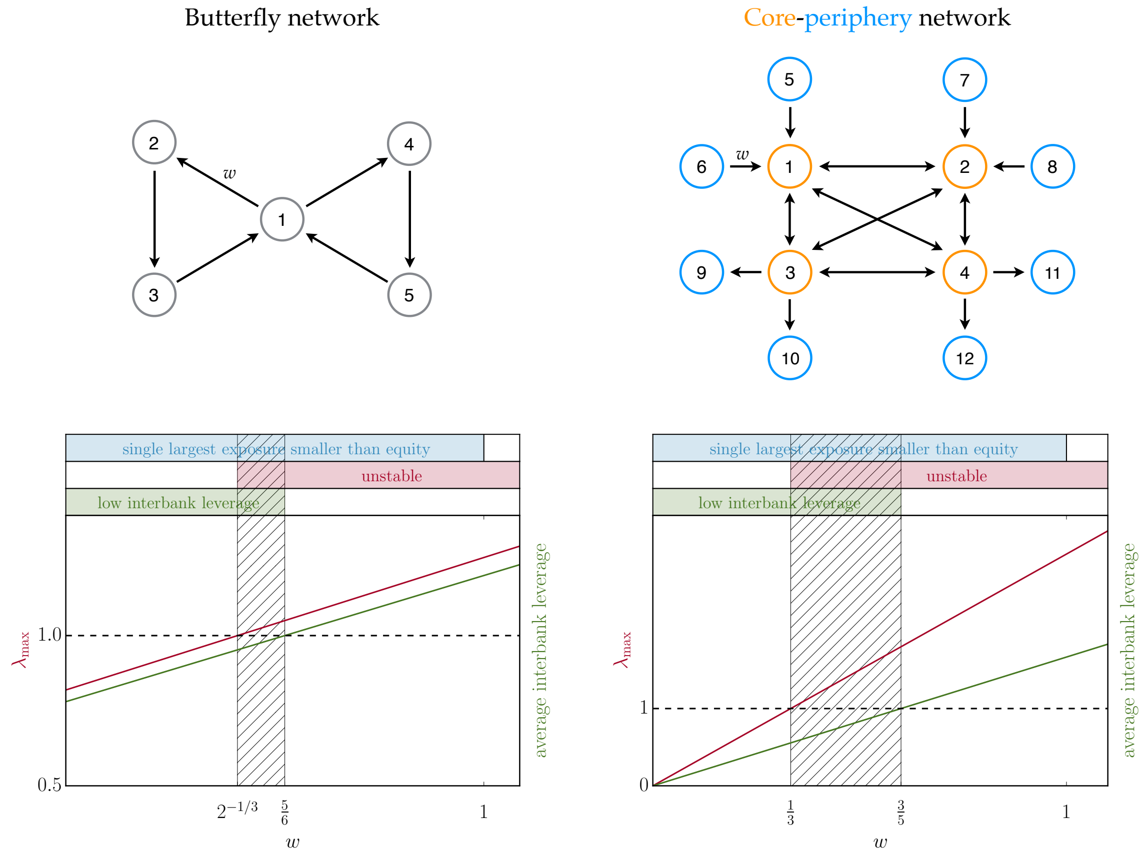

FIG. 1. For illustrative purposes we perform the stability analysis of the two paradigmatic interbank network architectures depicted above. The first is a "butterfly" graph, while the second example has a core-periphery topology: nodes 1, 2, 3, and 4 form a complete core, with the remaining nodes having either only incoming or outgoing edges to the core. For simplicity we set all non-zero elements of the interbank leverage matrix equal to w, implying that the largest single exposure policy is implemented whenever $w < 1$. We plot the average interbank leverage (green line) and $\lambda_{\max}$, the largest eigenvalue of the interbank leverage matrix (red line), as functions of the parameter w. The green region corresponds to an average interbank leverage smaller than one, the blue region to the largest single exposure smaller than the corresponding equity, while the unstable region is highlighted in red. In both cases there exists a region (shadowed in the figure) in which the following three properties hold: i) the average interbank leverage is larger than one, ii) the largest single exposure is smaller than the corresponding equity, and yet iii) the network is unstable. Slight modifications of the above examples can also account for tighter constraints on the largest single exposure. For example, even requiring that the largest single exposure is smaller than 15% of the equity (as requested in [26]) is not enough to avoid instability in a core-periphery topology with eight nodes in the core.

directed subgraph in which each node is reachable indirectly by any other). The Perron-Frobenius theorem only guarantees that the largest eigenvalue of a strongly connected component is between the minimum and the maximum interbank leverage across banks. Hence, a sufficient condition for instability (stability) is that the interbank leverage of all banks is larger (smaller) than one. However, for the years from 2008 to 2013, the smallest interbank leverage of European banks is very close to zero, while the 95th percentile of its distribution is between 2.5 and 6, meaning that the Perron-Frobenius bounds are not informative enough on the largest eigenvalue, and we need to look more closely at the topology of the network. For instance, for graphs without cycles (i.e. *directed acyclic graphs*, DAGs) $\lambda_{\max}$ is always equal to zero, implying that the presence of cycles is a necessary condition for instability (although not sufficient). Intuitively, a cycle amplifies distress propagation if the product of the weights of its edges is larger than one (we refer to this as an *individually unstable cycle*). Interestingly, a policy recommendation included in Basel III Accords [26] encourages banks to have the *largest single exposure* smaller than a fraction of their equity, so that $\Lambda_{ij} < 1$ for all i, j. The policy is thus effective in avoiding this source of instability.

However, the presence of individually unstable cycles, although sufficient, is not necessary for instability. Consider the two examples in Fig. 1. In particular, the second is a simple case of core-periphery network architecture, a frequently observed pattern in empirical interbank data

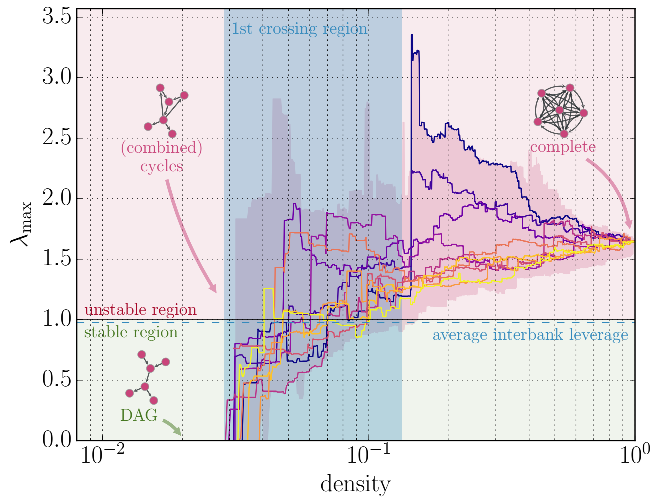

FIG. 2. Stability of the network of the top 50 listed European banks using data from their 2013 balance sheets. Trajectories have been built according to the following protocol: We start from a random DAG, i.e. a network with no cycles, which is therefore stable. We then increase the density of edges in the network by randomly adding one edge at a time, until the network is complete. Every time a new edge is added, we re-balance the interbank exposures so that the interbank leverages are always consistent with the real balance sheets. We compute $\lambda_{\max}$, the largest eigenvalue of the interbank leverage matrix, to assess the stability of the network: if $\lambda_{\max} > 1$ (< 1) the system is unstable (stable). We repeat the whole procedure 100 times. We show the contour of all trajectories and highlight a few of them. The first crossing region (in semi-transparent blue), spans the interval of densities of edges across which the networks become unstable for the first time, meaning that combined unstable cycles appear. We can see that densities as low as 3% are sufficient to reach instability. We also plot the average interbank leverage (dashed blue line) for reference.

[30]. In both cases, not only the largest single exposure policy is implemented, but (depending on the value of the parameter w) the average interbank leverage can be smaller than one. These two conditions could intuitively suggest that the system is stable. Yet, $\lambda_{\max}$ is larger than one and the system is unstable. The reason is that there are banks involved in multiple cycles. More precisely, a sufficient condition for having $\lambda_{\max} > 1$ is that there exist two integers i, k such that $(\Lambda^k)_{ii} > 1$, i.e. that there exists a bank i such that the sum, over all the cycles of length k from i to itself, of the products of the elements of the interbank leverage matrix along each of such cycles is larger than one (we refer to this as a *combined unstable cycle*). For instance, in the first example of Fig. 1, $(\Lambda^3)_{11}$ is larger than one for $w > 2^{-1/3}$, and thus there is a range of values where the system is unstable even if the largest single exposure policy is implemented and the average interbank leverage is smaller than 1.

The sufficient condition for instability stated above has important consequences for regulations intended to promote financial stability. Take the case of a bank having a given interbank leverage and at least one exposure larger than its equity. If now the bank is required to implement the largest single exposure policy and it wants to keep its interbank leverage unchanged, it might have to increase the number of its counterparties. On the one hand, this is beneficial because it reduces the exposures towards individual counterparties. On the other hand, it might be detrimental as it could contribute to the creation of new cycles that, even though might be individually stable, are part of a combined unstable cycle. Therefore, a recommendation that targets stability in terms of individual banks it can actually lead to instability because it neglects the systemic effect of cycles.

More in general, increasing the number of contracts in the system is the source of a second type of pathway towards instability. As a empirical illustration of this phenomenon, we consider the balance sheets of the top 50 listed banks in the European Union obtained from the Bankscope dataset. We simulate a process in which banks gradually increase the degree of risk diversification by continuously creating exposures towards additional counterparties. In particular, by adding edges, we build trajectories in the space of interbank networks whose initial configuration is a random DAG (hence stable) and whose final configuration is a complete graph. Every time a new edge is added, interbank exposures are redistributed so that the network is always consistent with the original balance sheets and interbank leverages of all banks do not change. For simplicity, we set the recovery rate ρ equal to zero. We find that the banking system is unstable once its graph is complete, but actually instability kicks in much earlier, when the fraction of existing contracts over all the possible ones is as low as 3% (see Fig. 2 for 2013 balance sheets, and SI Appendix for other years). Moreover, from Fig. 2 we see that trajectories of $\lambda_{\max}$ can be not monotonic and that the critical line can be crossed multiple times, meaning

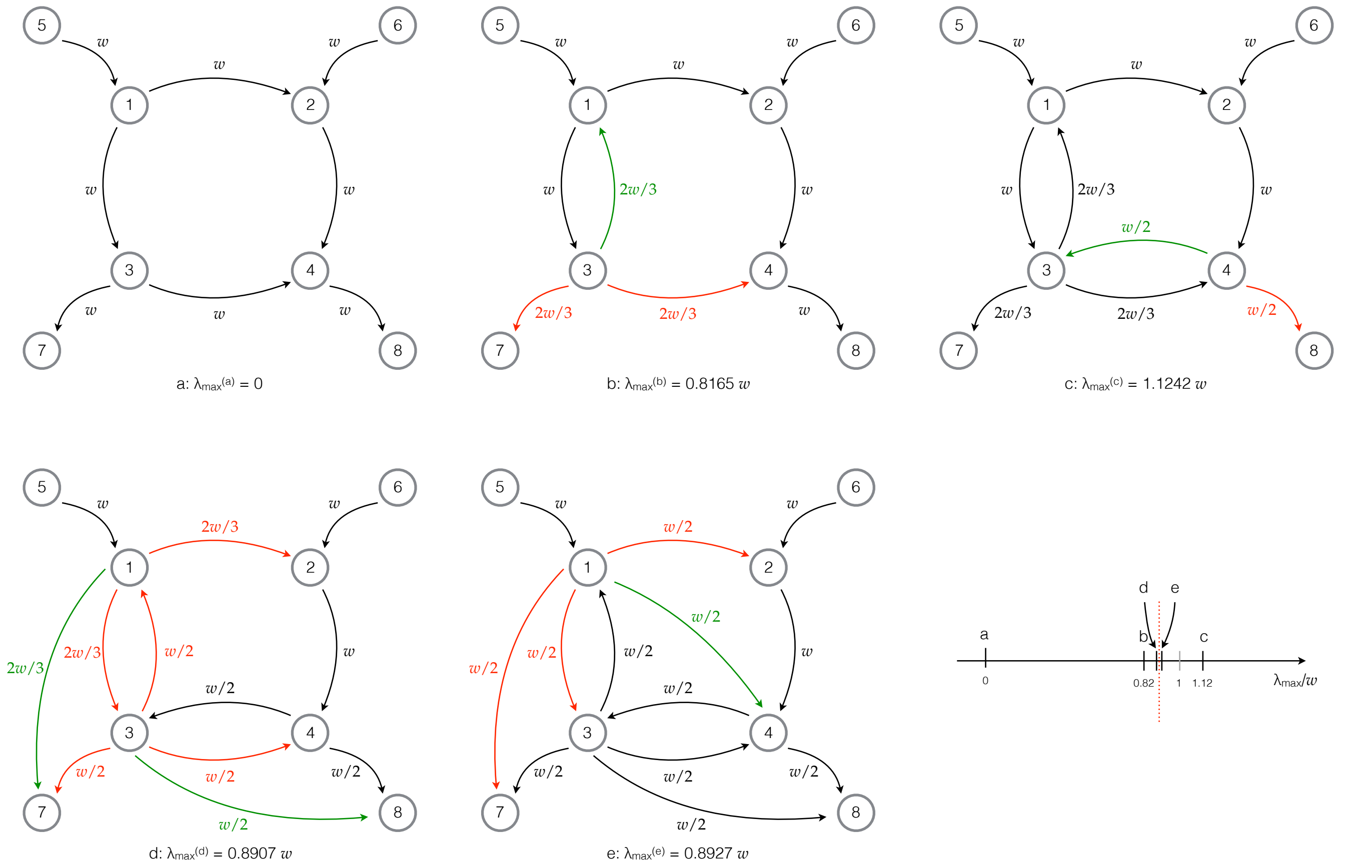

FIG. 3. Toy model of an interbank network that oscillates between stability and instability. Going from a to e we add one or more edges every time, always redistributing the weights so that interbank leverages do not change. Added edges are green, while modified edges are red. The initial network a is a DAG, hence $\lambda_{\max} = 0$, and for simplicity all edges have the same weight w. Suppose that w is chosen such that $\lambda_{\max}^{(d)} < 1 < \lambda_{\max}^{(e)}$. We then have that network b is stable, even tough a cycle has appeared. The further addition of one more cycle makes network c unstable. Network d becomes stable again after the addition of two edges, and finally network e is again unstable.

that the system sways between stability and instability, before finally settling into an unstable state.

In Fig. 3 we provide a stylized example that helps to connect such changes in the stability of the system to changes in the topology of the network. We start from a DAG, initially setting all non-zero elements of the interbank leverage matrix equal to w. We then add one edge at a time, always distributing the interbank leverage of each bank uniformly among the neighboring (borrowing) banks. $\lambda_{\max}$ increases every time a new cycle appears in the system. In contrast, $\lambda_{\max}$ decreases whenever a new edge does not lead to the appearance of a new cycle. Intuitively, this behavior can be explained in the following way. On the one hand, whenever a new cycle appears the possibility for the system to amplify shocks increases. On the other hand, whenever the addition of a new edge does not lead to the creation of a new cycle, the weights of those edges that are part of existing cycles become smaller because interbank leverages are constantly re-balanced, decreasing the ability of those cycles to amplify shocks.

CONCLUSIONS

By providing a simple and rigorous mathematical explanation of how network effects arise our results shed new light on the tension between the two main approaches to financial stability: the so-called microprudential one, focused on ensuring the stability of individual banks, and the macroprudential ones, targeted to the stability of the whole financial system.

We provide examples of sufficient conditions for the onset of instability: when banks establish contracts among each other without taking into account what their counterparties do, they will eventually become even unintentionally part of multiple cycles of contracts, which altogether amplify the effects of shocks. The recovery rate plays an important role, as it impacts directly the critical value of the largest eigenvalue. In turn, the recovery rate can be at least in part controlled with certain financial and monetary policies since it depends on both the quality of the collateral (in case of secured lending) and on the liquidity of the asset markets. Overall, our findings suggest that financial stability policies need to carefully consider network effects. This can be achieved by computing the largest eigenvalue of the interbank leverage matrix and by comparing it with estimates of the recovery rate.

More specifically, we show the existence of two processes that define trajectories in the space of network configurations, which drive financial networks from a stable to an unstable regime. The former consists of implementing processes of market integration (i.e. increasing the number of financial institutions) in a growing interbank network with interbank leverage larger than one. The latter consists of increasing the number of contracts among financial institutions. In both cases the risk profile of individual banks (measured by the interbank leverage) does not change, and therefore the emergence of instability is purely related to the structure of the network. This suggests that policies targeted at ensuring financial stability by lowering the risk of individual banks without taking into account the network effects can in fact lead to a higher systemic risk.

The relation between the topology of a network and the behaviour of dynamical processes taking place on the network is a fundamental problem common to a broad class of complex systems. Here we focus on financial contagion in a interbank network, but our results apply to any system in which the Jacobian of the dynamics is proportional to the weighted adjacency matrix (with non-negative entries) encoding the mutual interactions between the constituents of the system. In fact, our results expressed in terms of interbank leverages (sums over columns of the interbank leverage matrix) can be rephrased in terms of what in a more general setting are known as *strengths* of nodes (sums over columns of a weighted adjacency matrix). As a consequence, the existence of pathways to instability could in principle be observed in diverse contexts ranging from ecosystems to social networks.

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Supporting Information Appendix

LINEAR APPROXIMATION OF CONTAGION DYNAMICS

In this section we derive, under general mild assumptions, the linear approximation of distress propagation in interbank networks and we show how the stability of the system is related to the interbank leverage matrix. The first important ingredient is the balance-sheet consistency at all times t. The balance sheet of a bank is composed by assets and liabilities. The former have positive economic value (e.g. loans towards customers or towards other banks, stocks, derivatives, real estate), while the latter have negative economic value (e.g. deposits, debits towards other banks). In both cases, we distinguish between interbank and external assets or liabilities. Interbank assets (liabilities) are credits (debits) of banks towards other banks, while we call external all other assets and liabilities. We denote by $A_{ij}(t)$ the value at time t of a loan from bank i to bank j, and by $L_{ji}(t)$ the corresponding liability. External assets and liabilities of bank i at time t are denoted by $A_i^E(t)$ and $L_i^E(t)$, respectively. Finally, the equity $E_i(t)$ of bank i at time t is defined as the difference between its assets and liabilities:

$$E_i(t) = A_i^E(t) - L_i^E(t) + \sum_{j=1}^{n} A_{ij}(t) - L_{ij}(t)\,. \tag{S1}$$

We follow the assumption, common in the literature, that a bank defaults if its equity becomes negative. The rationale is that a bank with more debts that assets is not able to meet its obligations towards its creditors even by liquidating its entire pool of assets.

The second important ingredient is that the value of interbank assets change over time, as lenders constantly reassess their value based on the probability of default of their counterparties. Nevertheless, the actual amount of outstanding debt of borrowers does not change, i.e. liabilities always keep their face value. If bank j does not default its lender i will get $A_{ij}(0)$, the face value of the loan, while if bank j defaults i recovers the (smaller) amount R_{ij}. We assume that the value of interbank loans is assessed "fairly", i.e. it is equal to the expected repayment:

$$A_{ij}(t) = A_{ij}(0)(1 - p_j(t-1)) + R_{ij}p_j(t-1)\,, \tag{S2}$$

where $p_j(t)$ is the probability of default of bank j at time t. The time delay from the r.h.s. and the l.h.s. of (S2) accounts for the time needed for the information about the probability of default of borrowers to be incorporated into the assessment of lenders. Let us explicitly note that (S2) is consistent with assuming that $p_j(0) = 0$.

The scenario we have in mind is to initially stress the system via an exogenous shock to external assets, i.e. $A_i^E(0) \to A_i^E(1) < A_i^E(0)$. Balance sheet consistency (S1) implies that such shock will result in losses in equity. We assume that no additional cash flow (neither positive nor negative) enters the system subsequently. By assuming that the probability of default of a bank is an arbitrary function of its equity, equity losses will translate in changes in probabilities of default, and, via (S2), into a re-evaluation of interbank assets. This, in turn, will lead (again via (S1)) to a change in equity. In subsequent rounds external assets do not change and propagation of shocks continues only by iterating such dynamic through the interbank channel. As a consequence, two terms contribute to the loss in equity of bank i between time 0 to time t: the loss in external assets between time 0 and time 1 and the loss in interbank assets up to time t:

$$E_i(0) - E_i(t) = A_i^E(0) - A_i^E(1) + \sum_{j=1}^{n} \left[A_{ij}(0) - R_{ij}\right] p_j(t-1). \tag{S3}$$

By defining $h_i(t)$, the relative loss of equity at time t for bank i, as

$$h_i(t) = \frac{E_i(0) - E_i(t)}{E_i(0)}\,, \tag{S4}$$

and:

$$\hat{\Lambda}_{ij} = \frac{A_{ij}(0) - R_{ij}}{E_i(0)}\,, \tag{S5}$$

we can re-write (S3) as:

$$h_i(t) = h_i(1) + \sum_{j=1}^{n} \hat{\Lambda}_{ij} p_j(h_j(t-1))\,, \tag{S6}$$

where we also highlight the explicit dependence of the probability of default of bank j on its equity, and therefore on h_j. We stress that the assumptions made so far (balance sheets consistency, fair re-evaluation of interbank assets, probability of default as a generic function of the equity) can be considered accounting first principles.

By further assuming that $p_j(h_j(t))$ is differentiable in $h_j(0)$:

$$\begin{aligned} p_j(h_j(t)) &\simeq p_j(h_j(0)) + p_j'(h_j(0))\left[h_j(t) - h_j(0)\right] \\ &= p_j'(h_j(0))h_j(t)\,, \end{aligned} \tag{S7}$$

where we have used that $h_j(0) = 0$. By plugging (S7) into (S6) and by defining:

$$\tilde{\Lambda}_{ij} = \frac{A_{ij}(0) - R_{ij}}{E_i(0)} p_j'(h_j(0))\,, \tag{S8}$$

we can finally write

$$h_i(t) = h_i(1) + \sum_{j=1}^{n} \tilde{\Lambda}_{ij} h_j(t-1)\,. \tag{S9}$$

From the point of view of the stability of the system it is useful to compute $h_i(t+1) - h_i(t)$, which leads to:

$$h_i(t+1) = \min\left[1,\, h_i(t) + \sum_{j=1}^{n} \tilde{\Lambda}_{ij}\left[h_j(t) - h_j(t-1)\right]\right]\,, \tag{S10}$$

where the min simply ensures that relative equity losses cannot become larger than one, i.e. that equities do not become negative. However, equities stay positive in-between defaults and, by introducing $\Delta h(t) = h(t) - h(t-1)$, (S10) can be written in matrix form:

$$\Delta h(t+1) = \tilde{\Lambda}\Delta h(t)\,, \tag{S11}$$

which is a linear map with fixed point $\Delta h = 0$. The fixed point is stable if the modulus of the largest eigenvalue $\lambda_{\max}$ of $\tilde{\Lambda}$ is smaller than one. If $\lambda_{\max} > 1$, shocks will be amplified and at least one bank will default.

A further simplifying assumption is that R_{ij}, the amount recovered by the lender bank i in case of default of the borrower bank j, is a fraction ρ_j of the face value $A_{ij}(0)$:

$$R_{ij} = \rho_j A_{ij}(0)\,. \tag{S12}$$

From (S10) and (S12) we can see that ρ_j effectively dampens the losses suffered by lenders of bank j at all time steps, hence we call it *recovery rate*. Eqs. (S5) and (S8) become:

$$\hat{\Lambda}_{ij} = \Lambda_{ij}(1-\rho_j) \tag{S13a}$$

$$\tilde{\Lambda}_{ij} = \Lambda_{ij}(1-\rho_j)p_j'(h_j(0))\,, \tag{S13b}$$

where

$$\Lambda_{ij} = \frac{A_{ij}(0)}{E_i(0)}\,, \tag{S14}$$

is the *interbank leverage matrix*. From (S13b) we can make two observations. First, the structure of the network underlying $\tilde{\Lambda}$, i.e. the couple of banks between which there is an edge, is equal to the structure of the network underlying Λ. Second, a difference from bank to bank in the functions p_j is completely indistinguishable from heterogeneous recovery rates, by defining effective recovery rates one can absorb the factors $p_j'(h_j(0))$.

Choosing the equity loss as probability of default:

$$p_j(h_j((t)) = h_j(t)\,, \tag{S15}$$

implies that $p_j'(h_j(0)) = 1$, so that:

$$\tilde{\Lambda} = \hat{\Lambda} = \Lambda(1-\rho)\,, \tag{S16}$$

where ρ is a matrix whose j-th element on the diagonal is equal to ρ_j and with off-diagonal elements equal to zero. In this case (S7) is not an approximation, meaning that the dynamics (S10) and (S11) are exact and closely resemble the DebtRank [S1] linear dynamic $\Delta h(t+1) = \Lambda\Delta h(t)$. By performing the replacement $\tilde{\Lambda} \to \Lambda$ we obtain a generalized DebtRank that includes recovery.

ADDING NODES

The crucial thereom that we will exploit is due to Silverstein [S4] (Theorem 1.2). In a nutshell, let Λ be a $n \times n$ matrix whose entries are random i.i.d. variables with mean $\mu > 0$ and finite forth moment. For sufficiently large n, the largest eigenvalue $\lambda_{\max}$ of Λ is:

$$\lambda_{\max} = \frac{1}{n} \sum_{i,j} \Lambda_{ij} + \mathcal{O}(n^{-1/2}) \,. \tag{S17}$$

We will now specify the results of the theorem in the case in which the matrix Λ is the weighted adjacency matrix of a random graph. We consider Erdős-Renyi graphs in which $\Lambda_{ij} = C_{ij} W_{ij}$, with $C_{ij} \in \{0, 1\}$ and $W_{ij} \in \mathbb{R}^+$. The variables C_{ij} determine if an edge is present or not and have the bimodal distribution $\rho(C_{ij}) = p\delta(C_{ij} - 1) + (1 - p)\delta(C_{ij})$. The variables W_{ij} are the weights associated to the edges and we leave their distribution unspecified (as long as the forth moment is finite).

We start with the case in which the network is *not* sparse, i.e. the case in which the average degree $\bar{k} \equiv \sum_{ij} C_{ij}/n$ is $\bar{k} \simeq \mathcal{O}(n)$, or equivalently $p \simeq \mathcal{O}(1)$ (in the sense that it does not scale with n). Let us define the variables X_i, $i = 1, \ldots n$, as the sums only over columns of Λ, i.e. $X_i = \sum_j C_{ij} W_{ij}$. As C_{ij} and W_{ij} are independent, we have:

$$\langle X_i \rangle = n \langle C_{ij} \rangle \langle W_{ij} \rangle = np \langle W_{ij} \rangle \tag{S18a}$$

$$\operatorname{var} X_i = n \operatorname{var}(C_{ij} W_{ij}) = n \left[p \langle W_{ij}^2 \rangle - p^2 \langle W_{ij} \rangle^2 \right] \,. \tag{S18b}$$

The next step is to compute $\sum_i X_i/n$. As X_i are i.i.d. with finite variance, using (S17) we have that $\lambda_{\max}$ will be normally distributed with

$$\langle \lambda_{\max} \rangle = \frac{1}{n} n \langle X_i \rangle = np \langle W_{ij} \rangle \tag{S19a}$$

$$\operatorname{var} \lambda_{\max} = \frac{1}{n^2} n \operatorname{var} X_i = \left[p \langle W_{ij}^2 \rangle - p^2 \langle W_{ij} \rangle^2 \right] \,, \tag{S19b}$$

meaning that the relative fluctuation is $\sqrt{\operatorname{var} \lambda_{\max}}/\langle \lambda_{\max} \rangle \simeq 1/n$.

In the case in which the graph is sparse, i.e. $\bar{k} \simeq \mathcal{O}(1)$ and $p \simeq 1/n$ we know that the degree of each node has a Poisson distribution with mean $\bar{k}$. As a consequence, X_i will have a compound Poisson distribution with

$$\langle X_i \rangle = \bar{k} \langle W_{ij} \rangle \tag{S20a}$$

$$\operatorname{var} X_i = \bar{k} \langle W_{ij}^2 \rangle \,. \tag{S20b}$$

If we now compute the first two moments of $\sum_i X_i/n$ we find that:

$$\langle \lambda_{\max} \rangle = \frac{1}{n} n \langle X_i \rangle = \bar{k} \langle W_{ij} \rangle \tag{S21a}$$

$$\operatorname{var} \lambda_{\max} = \frac{1}{n^2} n \operatorname{var} X_i = \frac{\bar{k} \langle W_{ij}^2 \rangle}{n} \,, \tag{S21b}$$

meaning that the relative fluctuation is $\sqrt{\operatorname{var} \lambda_{\max}}/\langle \lambda_{\max} \rangle \simeq 1/\sqrt{n}$. Moreover, we can see that the fluctuation on $\langle \lambda_{\max} \rangle$ is of the same order of the correction in (S17), therefore we are not able to compute the distribution of $\lambda_{\max}$ in this case.

In the previous derivation we assumed that all entries of the interbank leverage matrix are i.i.d., which is not entirely true. In fact, in our networks a bank cannot extend a loan to itself, meaning that there are no loops (cycles of length one), i.e. the diagonal of the weighted adjacency matrix is filled with zeros. To compute the relative correction on $\langle \lambda_{\max} \rangle$ it will suffice to note that if λ is an eigenvalue of a matrix M, $\lambda - a$ is an eigenvalue of the matrix $M - a\mathbb{I}$. As a consequence, in the case of sparse graphs, we have that $\langle \lambda_{\max} \rangle = np \langle W_{ij} \rangle - p \langle W_{ij} \rangle = (n - 1)p \langle W_{ij} \rangle$. Since for graphs without loops $\bar{k} = (n - 1)p$, we have that $\langle \lambda_{\max} \rangle = \bar{k} \langle W_{ij} \rangle$. In the case of sparse graphs the correction is already accounted for in (S21a), provided that the correct value of $\bar{k}$ is used.

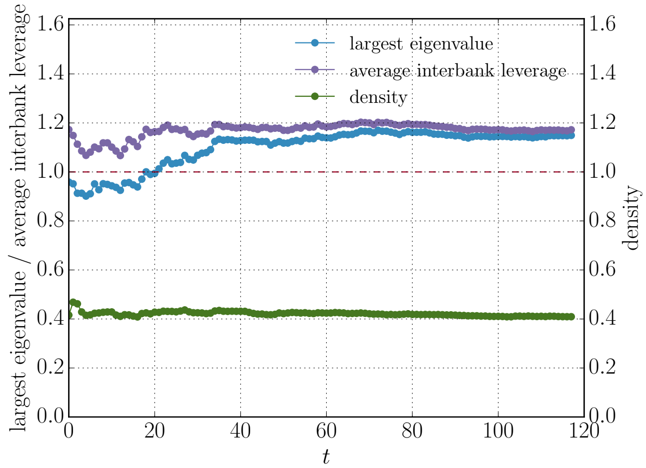

FIG. S1. Example of growth process in which a stable network with average interbank leverage larger than one becomes unstable as new banks are added to the system. We stress that the crossing to the unstable regime is genuinely driven by the fact that fluctuations in the asymptotic distribution of $\lambda_{\max}$ shrink as n becomes larger: in fact the density of edges in the network stays roughly constant. Here the initial network has $n = 20$ and the weight distribution is exponential.

In both cases we have that $\lambda_{\max} = \bar{k}\langle W_{ij}\rangle$, as $n \to \infty$, but with different relative fluctuations. It is worth noting that, when Λ is the matrix of interbank leverage, $\bar{k}\langle W_{ij}\rangle$ is precisely the average interbank leverage ℓ. Therefore, for $n \to \infty$, if $\ell > 1$ the system will be unstable, while if $\ell < 1$ it will be stable. However, if n is not large, fluctuations are relevant, and a system can be stable even if $\ell > 1$, and vice versa. We now provide an example of how adding nodes to such a network can make the system unstable. We start by randomly generating an Erdős-Renyi graph with given p and using an exponential distribution of weights with mean $\langle W_{ij}\rangle$, so that $\ell > 1$, stopping as soon as we find a stable graph. We then proceed to add a new node at a time, by preserving the property that all entries of the weighted adjacency matrix are i.i.d. and by keeping the density of edges (i.e. $\bar{k}$) constant. In fact, if we devised a growth process in which $\bar{k}$ increases, the system would trivially become unstable. We use the following algorithm. Let n be the number of nodes before the addition of a new node i. (i) We randomly form edges from node i and each of the other n nodes with probability p; (ii) we draw a weight from the weight distribution for each of the new outcoming edges from i: (iii) we rescale such weights multiplying them by $(n-1)/n$; (iv) we randomly form edges from each of the other n nodes to node i with probability p; (v) we draw a weight from the weight distribution for each of the new incoming edges for i; (vi) we rescale the weights of all edges starting from the new neighbors of i (including the ones towards node i) so that the sum of all weights of the edges coming out from those nodes do not change after the addition of node i. In Fig. S1 we see a realization of such process in which both the density of edges and the average interbank leverage are roughly constant, while $\lambda_{\max}$ becomes larger than one, driving the system towards the instability. Let us note that such algorithm is designed to keep all interbank leverages constant. However, the probability distribution of single entries of the interbank leverage matrix may vary from a step of the algorithm to the next one. We have checked that the simpler variant in which one keeps the probability distribution of single entries constant and the interbank leverage constant only on average yields the same results.

ADDING EDGES

For completeness, we show in Fig. S2 here the analogous of Fig. 3 of the main text for the years between 2008 and 2012. We also include the year 2013 (already in Fig. 3 of the main text) for reference.

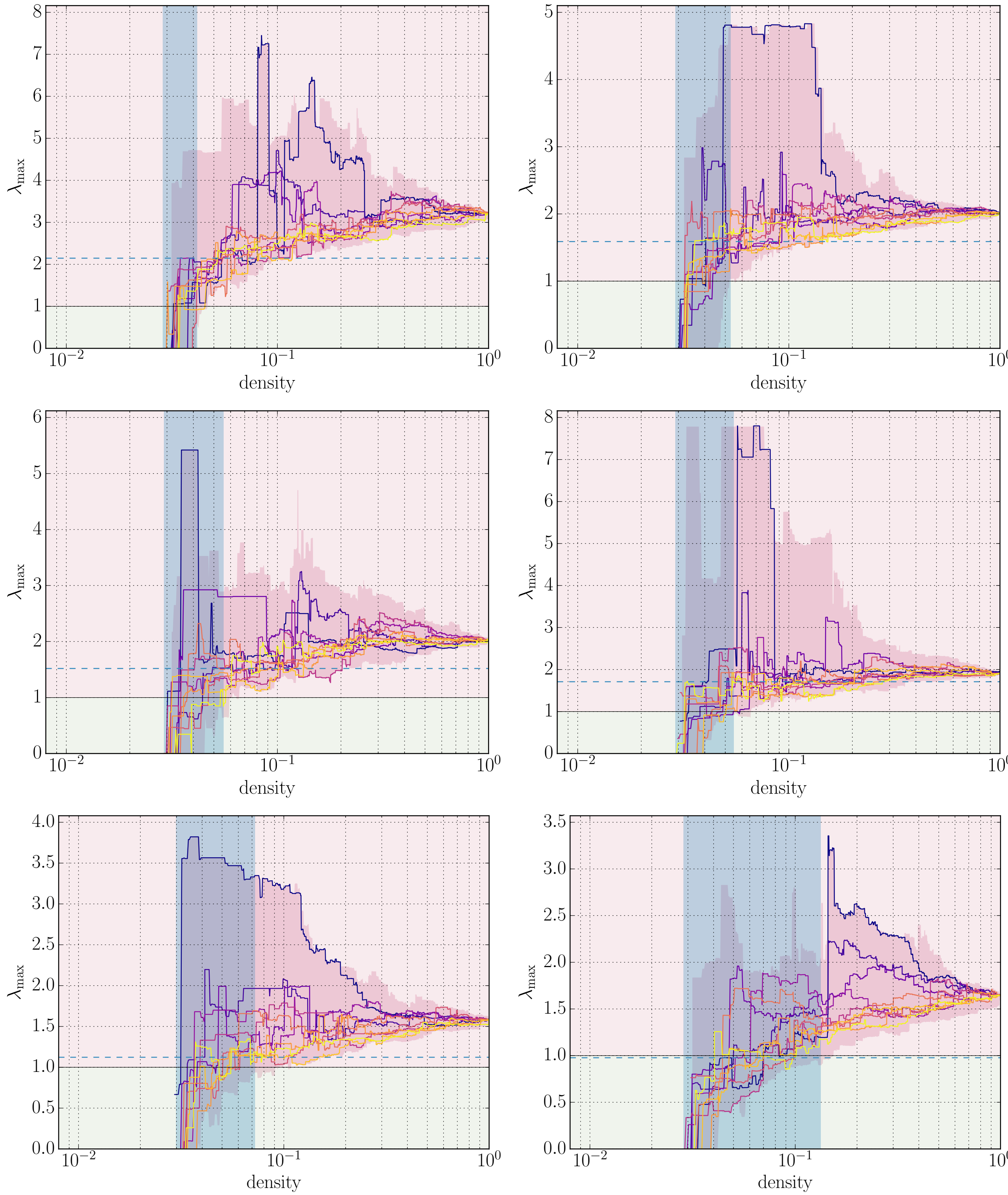

FIG. S2. Analogous of Fig. 3 of the main text for years from 2008 to 2013. 2008: upper left, 2009: upper right, 2010: mid left, 2011: mid right, 2012: bottom left, 2013: bottom right. For $\lambda_{\max} < 1$ the interbank network is stable (green region), while for $\lambda_{\max} > 1$ it is unstable (red region). For comparison we also plot (dashed blue line) the average interbank leverage.

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