CENTRAL LIMIT THEOREMS FOR A HYPERGEOMETRIC RANDOMLY REINFORCED URN

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Abstract. We consider a variant of the randomly reinforced urn where more balls can be simultaneously drawn out and balls of different colors can be simultaneously added. More precisely, at each time-step, the conditional distribution of the number of extracted balls of a certain color given the past is assumed to be hypergeometric. We prove some central limit theorems in the sense of stable convergence and of almost sure conditional convergence, which are stronger than convergence in distribution. The proven results provide asymptotic confidence intervals for the limit proportion, whose distribution is generally unknown. Moreover, we also consider the case of more urns subjected to some random common factors.

1. Introduction

Urn models, also known as preferential attachment models, are stochastic processes in which, along the time-steps, different individuals or objects or categories (represented by different colors) receive some quantity, called "weight" (represented by the number of balls), in such a way that the higher the total weight they already have until a certain time, the greater the probability of receiving an additional weight at the next time (i.e. a "self-reinforcing" property). The preferential attachment is a key feature governing the dynamics of many biological, economic and social systems. Therefore, urn models are a very popular topic because of their hints for theoretical research and their applications in various fields: clinical trials (e.g. [5, 22, 25, 31, 37, 45]), economics and finance (e.g. [7, 27, 30]), information science (e.g. [35, 36]), network theory (e.g. [14, 16, 20]) and so on.

The first example of urn scheme is the standard Eggenberger-Pólya urn [26, 41]: an urn contains a red and b black balls and, at each discrete time, a ball is drawn out from the urn and then it is put again inside the urn together with an additional constant number $k > 0$ of other balls of the same color. Let Z_n be the proportion of red balls at time n, namely, the conditional probability of drawing a red ball at time $n + 1$, given the outcomes of the previous extractions. A well known result (see, for instance, [35]) states that (Z_n) is a bounded martingale and Z_n converges almost surely to a random variable Z with Beta distribution with parameters a/k and b/k.

Subsequently, urn models have been widely studied by many researchers and there is a rather extensive literature on them (e.g. [2, 4, 9, 10, 15, 17, 18, 24, 33, 34, 38, 44]): a large number of new "replacement policies" (for instance, balanced rules,

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tenable mechanisms and random reinforcements) and various related models (for instance, the Poisson-Dirichlet model [6] and the very recent Indian buffet model [8]) have been introduced and analyzed from different points of view and by means of different techniques (combinatorial methods, martingales, branching processes, stochastic approximations, etc.). We refer to [40], and the references therein, for a general survey on random processes with reinforcement.

In particular, as an extension of the Pólya urn, the Randomly Reinforced Urn (RRU) was recently proposed and analyzed [2, 9, 10, 11, 12, 21, 22, 37, 38, 43, 44]. It consists in a multicolor urn which is reinforced at each time with a random number of additional balls according to the color of the extracted ball. The distribution of the reinforcements may depend on time and be different for the different colors. These models are suitable in order to describe the evolution of some system, such as a population, and also to perform an adaptive design, i.e. an experimental design that uses accumulated data to decide on how to carry on the study, without undermining the validity and the integrity of the experiment. Indeed, the RRU model provides randomized treatment allocation schemes (clinical trials) where patients are assigned to the best treatment with probability converging to one [10, 37].

In [3] a new version of the RRU model is formulated. This model consists of an urn which contains balls of two different colors, say $a \in \mathbb{N} \setminus \{0\}$ balls of color A and $b \in \mathbb{N} \setminus \{0\}$ balls of color B. At each time $n \geq 1$, we simultaneously (i.e. without replacement) draw a random number N_n of balls. Let X_n be the number of extracted balls of color A. Then we return the extracted balls in the urn together with other $R_n X_n$ balls of color A and $R_n(N_n - X_n)$ balls of color B. The size R_n of the reinforcement is assumed independent of $[N_1, X_1, R_1, \ldots, N_{n-1}, X_{n-1}, R_{n-1}, N_n, X_n]$. We will call this model "Hypergeometric Randomly Reinforced Urn" (HRRU).

With respect to the RRU model, the main novelties of this model are that, at each time, more balls can be simultaneously drawn out (and returned in the urn) and balls of different colors can be simultaneously added. The number of extracted balls of a certain color depends on the composition of the urn at the moment of the extraction, akin a preferential attachment rule. When $N_n = 1$ for each n, the HRRU reduces to the RRU model with equal reinforcements for the two colors. In particular, the case $N_n = 1$ and $R_n = k$ (where k is a constant) for each n corresponds to the standard Eggenberger-Pólya urn; while the case $N_n = h$ and $R_n = k$ (where h and k are two constants) for each n coincides with the model in [18, 19]. Also the model introduced and studied in [22] can be seen as a RRU model where balls of different colors can be simultaneously added, but there the "multi-updating" is due to a delay in the updating. Indeed, at each time n a single ball is drawn out (and returned in the urn) but the updating is performed at certain time-steps $(u_i)_{i \geq 1}$ as follows: at time u_i, we add a random number R_n of balls of the same color of the ball extracted at time n, for each $n = u_{i-1} + 1, \ldots, r_i$, with $r_i \leq u_i$.

As explained in [3], a possible interpretation of the HRRU model is the following. At each time $n \geq 1$, a new firm appears on the market and it has to choose the operative system for its computers among two different types, say operative system A (to which we associate color A) and operative system B (to which we associate color B). The total number of its computers is $R_n N_n$ (more precisely, N_n blocks of R_n computers each). The firm decides to adopt X_n blocks (of size R_n each) with operative system A and $(N_n - X_n)$ blocks (of size R_n each) with operative systems

B, according to the number of computers with operative systems A already present in the market. Another possible interpretation follows. At each time $n \geq 1$, a pharmaceutical firm has to select the size of its production for two different kinds of products, say product A and product B. For instance, A and B can be two medicines for the same disease but with different costs. The total of its production is $R_n N_n$ (more precisely, the firm produces N_n blocks, each of size R_n). The firm decides to produce X_n blocks of type A-products and $(N_n - X_n)$ blocks of type B-products according to the number of type A-products already on the market. Finally, setting $R_n = 1$ for each n, the HRRU model can be employed to describe the growth of a population in which we can distinguish two types of individuals, say A and B. At each time n, the random numbers N_n and X_n represent the new offsprings and the new offsprings of type A, respectively. The number of the new type A-individuals depends on the composition of the population at the preceeding time-step.

It is shown in [3], that Z_n converges almost surely to a random variable Z, whose distribution is generally unknown. Authors also provide some results concerning the distribution of the limit random variable Z in some particular cases. In the present paper we continue the study of the model proving some central limit theorems and making another step toward the description of the distribution of Z. Further, the proven central limit theorems can be used in order to obtain asymptotic confidence intervals for the limit proportion Z. Moreover, we can also consider the case of more urns (for instance, according to the previous interpretations, the different urns can represent different markets or different populations), each of them following a HRRU dynamics, and perform some test for comparing them or get asymptotic confidence intervals for any linear combination of the limit proportions.

The paper is organized as follows. In Section 2 we formally introduce the model. In Section 3 we recall the needed facts concerning stable convergence and almost sure conditional convergence. In Section 4 we give and discuss the main results, whose proofs are postponed to Section 5. Finally, in Section 6 we provide some statistical tools based on the proven results. The paper is enriched with an appendix which contains some useful auxiliary results.

2. The HRRU model

An urn contains $a \in \mathbb{N} \setminus \{0\}$ balls of color A and $b \in \mathbb{N} \setminus \{0\}$ balls of color B. At each time $n \geq 1$, we simultaneously (i.e. without replacement) draw a random number N_n of balls. Let X_n be the number of extracted balls of color A. Then we return the extracted balls in the urn together with other $R_n X_n$ balls of color A and $R_n(N_n - X_n)$ balls of color B. More precisely, we take a probability space $(\Omega, \mathcal{A}, P)$ and, on it, some random variables N_n, X_n, R_n such that, for each $n \geq 1$, we have:

i) The conditional distribution of the random variable N_n given

$$[N_1, X_1, R_1, \dots, N_{n-1}, X_{n-1}, R_{n-1}]$$

is concentrated on $\{1, \dots, S_{n-1}\}$ where

$$S_{n-1} = a + b + \sum_{j=1}^{n-1} N_j R_j = \text{total number of balls at time } n-1. \tag{1}$$

ii) The conditional distribution of the random variable X_n given

$$[N_1, X_1, R_1, \dots, N_{n-1}, X_{n-1}, R_{n-1}, N_n]$$

is hypergeometric with parameters N_n, S_{n-1} and H_{n-1}[1] where

$$H_{n-1} = a + \sum_{j=1}^{n-1} X_j R_j = \text{total number of balls of color A at time } n-1. \tag{2}$$

iii) The random variable R_n takes values in $\mathbb{N} \setminus \{0\}$ and it is independent of

$$[N_1, X_1, R_1, \dots, N_{n-1}, X_{n-1}, R_{n-1}, N_n, X_n].$$

Note that we do not specify the conditional distribution of N_n given the past $[N_1, X_1, R_1, \dots, N_{n-1}, X_{n-1}, R_{n-1}]$ nor the distribution of R_n.

We will refer to the above urn model as the *Hypergeometric Randomly Reinforced Urn* (HRRU)[2]. It is worthwhile to remark that this model include the classical Pólya urn (the case with $N_n = 1$ and $R_n = k$ for each n) and the randomly reinforced urn with the same reinforcements for both colors (the case with $N_n = 1$ for each n and R_n arbitrarily random).

We set Z_n equal to the proportion of balls of color A in the urn (immediately after the updating of the urn at time n and immediately before the $(n+1)$-th extraction), that is $Z_0 = a/(a+b)$ and

$$Z_n = \frac{H_n}{S_n} \quad \text{for } n \geq 1.$$

Moreover we set

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma\big(N_1, X_1, R_1, \dots, N_n, X_n, R_n\big) \quad \text{for } n \geq 1\,,$$

and

$$\mathcal{G}_n = \mathcal{F}_n \vee \sigma(N_{n+1}), \quad \mathcal{H}_n = \mathcal{G}_n \vee \sigma(R_{n+1}) \quad \text{for } n \geq 0.$$

3. Stable convergence and almost sure conditional convergence

Stable convergence has been introduced by Rényi in [42] and subsequently investigated by various authors, e.g. [1, 23, 28, 32, 39]. It is a strong form of convergence in distribution, in the sense that it is intermediate between the simple convergence in distribution and the convergence in probability. In this section we recall some basic definitions and properties. For more details, we refer the reader to [23, 29] and the references therein.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let S be a Polish space (i.e. a completely metrizable separable topological space), endowed with its Borel σ-field. A *kernel* on S, or a random probability measure on S, is a collection $K = \{K(\omega, \cdot) : \omega \in \Omega\}$

[1] We recall that a random variable X has hypergeometric distribution with parameter N, S, H if $P\{X = k\} = \frac{\binom{H}{k}\binom{S-H}{N-k}}{\binom{S}{N}}$

[2] It coincides with the model introduced in [3] but here the adopted notation is different: M_n in [3] corresponds to our N_n (total number of extracted balls at time n), R_n in [3] corresponds to our X_n (number of extracted balls of color A at time n) and N_n in [3] corresponds to our R_n (number of added balls for each extracted ball at time n). We decided to adopt a different notation with respect to [3] in order to use a notation more similar to the one used in the RRU model literature.

of probability measures on the Borel σ-field of S such that, for each bounded Borel real function f on S, the map

$$\omega \mapsto Kf(\omega) = \int f(x)\, K(\omega, dx)$$

is $\mathcal{A}$-measurable. Given a kernel on S and an event H in $\mathcal{A}$ with $P(H) > 0$, we can define a probability measure on S, denoted by $P_H K$, as follows:

$$P_H K(B) = E[K(\cdot, B)|H] = P(H)^{-1} \int_H K(\omega, B)\, P(d\omega),$$

for each Borel set B of S. We simply write PK when $H = \Omega$. It is easy to verify the relation

$$\int f(x)\, P_H K(dx) = P(H)^{-1} \int_H Kf(\omega)\, P(d\omega).$$

On $(\Omega, \mathcal{A}, P)$ let (Y_n) be a sequence of S-valued random variables and let K be a kernel on S. Then we say that Y_n converges *stably* to K, and we write $Y_n \overset{stably}{\longrightarrow} K$, if

$$P(Y_n \in \cdot \,|\, H) \overset{weakly}{\longrightarrow} P_H K \qquad \text{for all } H \in \mathcal{A} \text{ with } P(H) > 0.$$

Clearly, if $Y_n \overset{stably}{\longrightarrow} K$, then Y_n converges in distribution to the probability measure PK. Moreover, we recall that the convergence in probability of Y_n to a random variable Y is equivalent to the stable convergence of Y_n to a special kernel, which is the Dirac kernel $K = \delta_Y$.

We next mention a form of convergence, called almost sure conditional convergence, introduced and studied in [21], and afterwards employed by other researchers (see, for example, [2, 43]).

For each n, let $\mathcal{F}_n$ be a sub-σ-field of $\mathcal{A}$ and set $\mathcal{F} = (\mathcal{F}_n)$ (called conditioning system). If K_n denotes a version of the conditional distribution of Y_n given $\mathcal{F}_n$, we say that Y_n converges to K in the sense of the *almost sure conditional convergence* with respect to $\mathcal{F}$, if, for almost every ω in Ω, the probability measure $K_n(\omega, \cdot)$ converges weakly to $K(\omega, \cdot)$. Evidently, if Y_n converges to K in the sense of the almost sure conditional convergence with respect to $\mathcal{F}$, we have that

$$E\,[f(Y_n)\,|\,\mathcal{F}_n] \overset{a.s.}{\longrightarrow} Kf$$

for each bounded continuous real function f on S and Y_n converges in distribution to the probability measure PK.

In the sequel we will adopt the notation $\mathcal{N}(0, V)$ in order to indicate the *Gaussian kernel* with zero mean and random variance V, that is the collection $\{\mathcal{N}(0, V(\omega)) : \omega \in \Omega\}$ of centered Gaussian distributions, where V is a positive random variable ($\mathcal{N}(0, 0)$ is meant as the Dirac probability measure concentrated in zero). Further, given two kernels K_1 and K_2, we will denote by $K_1 \otimes K_2$ the kernel given by the product measures $K_1(\omega, \cdot) \otimes K_2(\omega, \cdot)$.

4. Convergence results for the HRRU model

The sequence (Z_n) is a bounded $\mathcal{H}$-martingale. Indeed, we have

$$Z_n - Z_{n-1} = \frac{R_n(X_n - N_n Z_{n-1})}{S_n} \tag{3}$$

and so

$$E[Z_n - Z_{n-1}|\mathcal{H}_{n-1}] = \frac{R_n}{S_n}\left(E[X_n|\mathcal{H}_{n-1}] - N_n Z_{n-1}\right) = \frac{R_n}{S_n}\left(E[X_n|\mathcal{G}_{n-1}] - N_n Z_{n-1}\right)$$
$$= 0$$

(where the second equality holds true because of condition iii) and the last one is implied by condition ii)). Hence, the sequence (Z_n) converges almost surely (and in L^1) to a random variable Z. Lemma A.2 (with $Y_n = X_n/N_n$) immediately implies that the sequence

$$M_n = \frac{1}{n}\sum_{j=1}^{n}\frac{X_j}{N_j} \tag{4}$$

also converges almost surely (and in L^1) to Z (cfr. Th. 3.1, Th. 3.5 in [3]).

The distribution of Z is unknown except in a few particular cases (see [3]). We are going to prove the following central limit theorems, useful in order to get some information on Z.

Theorem 1. *Assume there exists a constant $k \in \mathbb{N} \setminus \{0\}$ such that $N_n \vee R_n \leq k$ for each n and*

$$E[N_n|\mathcal{F}_{n-1}] \xrightarrow{a.s.} N, \quad E[R_n] \longrightarrow m, \quad E[R_n^2] \longrightarrow q\,, \tag{5}$$

where N is a strictly positive bounded random variable and m and q are finite and strictly positive numbers.

Then $\sqrt{n}(Z_n - Z)$ converges in the sense of the almost sure conditional convergence with respect to $\mathcal{F} = (\mathcal{F}_n)$ to the Gaussian kernel $\mathcal{N}(0, V)$, where

$$V = qm^{-2}N^{-1}Z(1-Z).$$

Theorem 2. *Under the assumptions of Theorem 1, suppose also that*

$$E[N_n^{-1}|\mathcal{F}_{n-1}] \xrightarrow{a.s.} \eta, \tag{6}$$

where η is a strictly positive bounded random variable.

Then

$$[\sqrt{n}(M_n - Z_n), \sqrt{n}(Z_n - Z)] \xrightarrow{stably} \mathcal{N}(0, U) \otimes \mathcal{N}(0, V),$$

where

$$U = V + Z(1-Z)\left(\eta - 2N^{-1}\right) = \left(qm^{-2}N^{-1} + \eta - 2N^{-1}\right)Z(1-Z).$$

From the above theorems we have that $\sqrt{n}(M_n - Z_n)$ converges stably to $\mathcal{N}(0, U)$ and $\sqrt{n}(M_n - Z)$ converges stably to $\mathcal{N}(0, U + V)$.

The following corollary enriches Corollary 3.4 in [3].

Corollary 3. *Assume there exists a constant $k \in \mathbb{N} \setminus \{0\}$ such that $N_n \vee R_n \leq k$ for each n. Then:*

a) $P(Z = 0) + P(Z = 1) < 1.$

b) *If assumptions (5) are also satisfied, then* $P(Z = z) = 0$ *for all* $z \in (0, 1)$.

Note that the above result entails that the limit Gaussian kernel in Theorem 1 is not degenerate.

Some examples and comments follow.

Example 4. If $N_n = h_n$ with $h_n \in \mathbb{N} \setminus \{0\}$ and $h_n \uparrow h \leq a + b$, then the first condition in (5) and condition (6) are obviously satisfied with $N = h$ and $\eta = h^{-1}$, so that we have $V = qm^{-2}h^{-1}Z(1 - Z)$ and $U = (qm^{-2} - 1)h^{-1}Z(1 - Z)$.

Remark 5. If (N_n) is a sequence of integer-valued random variables with $1 \leq N_n \leq k$ and converging almost surely to a random variable N, then (by Lemma A.1) the first condition in (5) holds true. Moreover, condition (6) is satisfied with $\eta = N^{-1}$ and so we have $U = (qm^{-2} - 1)N^{-1}Z(1 - Z)$.

The next example concerns the above remark.

Example 6. Suppose that (N_n) is given by a symmetric random walk with two absorbing barriers. More precisely, given $h \in \mathbb{N}$, with $2 \leq h \leq a + b$, set

$$\widetilde{N}_1 = i \in \{2, \ldots, h - 1\}, \qquad \widetilde{N}_n = i + \sum_{j=1}^{n-1} Y_j$$

where each Y_j is independent of $[X_1, R_1, Y_1, X_2, R_2, \ldots, Y_{j-1}, X_j, R_j]$ and such that $P(Y_j = -1) = P(Y_j = 1) = 1/2$. Set $\Gamma_1 = 0$ and $\Gamma_n = \sum_{j=1}^{n-1} Y_j$ for $n \geq 2$, and define

$$T_1 = \inf\{n : \widetilde{N}_n = 1\} = \inf\{n : \Gamma_n = 1 - i\}$$
$$T_h = \inf\{n : \widetilde{N}_n = h\} = \inf\{n : \Gamma_n = h - i\}.$$

Finally, for each $n \geq 1$, set $N_n = \widetilde{N}_{T \wedge n}$ where $T = T_1 \wedge T_h$. Then $N_n \xrightarrow{a.s.} N = \widetilde{N}_T$ where $N = I_{\{T=T_1\}} + hI_{\{T=T_h\}}$. In order to find the probabilities $P(T = T_1) = p$ and $P(T = T_h) = 1 - p$, it is enough to observe that, since (Γ_n) is a martingale, we have

$$E[\Gamma_T] = (1 - i)p + (h - i)(1 - p) = 0$$

and so $p = (h - i)/(h - 1)$. According to Remark 5, $\eta = N^{-1} = I_{\{T=T_1\}} + h^{-1}I_{\{T=T_h\}}$.

The last example regards the case when the random variables N_n are independent and identically distributed.

Example 7. Suppose that (N_n) are a sequence of random variables such that each N_n is independent of $\mathcal{F}_{n-1}$ and uniformly distributed on the set $\{1, \ldots, h\}$, with $2 \leq h \leq a + b$. Then $N = E[N_n] = (h + 1)/2$ and $\eta = E[N_n^{-1}] = h^{-1} \sum_{j=1}^{h} j^{-1}$.

5. Proofs

We begin with a preliminary result.

Proposition 8. *Assume there exists a constant* $k \in \mathbb{N} \setminus \{0\}$ *such that* $N_n \vee R_n \leq k$ *for each* n *and*

$$E[N_n|\mathcal{F}_{n-1}] \xrightarrow{a.s.} N, \quad E[R_n] \longrightarrow m, \tag{7}$$

where N is a strictly positive bounded random variable and m is a finite and strictly positive number.

Then

$$\frac{S_n}{n} \xrightarrow{a.s.} Nm.$$

Proof. It follows from Lemma A.2 with $Y_j = N_j R_j$. Indeed, we have $Y_j^2 \leq k^4$ for each j and (by iii))

$$E[N_j R_j | \mathcal{F}_{j-1}] = E[N_j | \mathcal{F}_{j-1}] E[R_j] \xrightarrow{a.s.} Nm\,.$$

□

Proof of Theorem 1. Setting $X'_n = X_n / N_n$ for each n, the sequence (X'_n) is $\mathcal{G}$-adapted and bounded. Moreover, we have

$$E[X'_{n+1}|\mathcal{G}_n] = E[N_{n+1}^{-1} X_{n+1}|\mathcal{G}_n] = N_{n+1}^{-1} E[X_{n+1}|\mathcal{G}_n] = N_{n+1}^{-1} N_{n+1} Z_n = Z_n \tag{8}$$

and, as we have already said, the sequence (Z_n) is a bounded $\mathcal{G}$-martingale. Therefore, in order to prove Theorem 1, it suffices to prove that the following conditions are satisfied (see Theorem A.3 applied to $Y_n = X'_n$):

c1) $E[\sup_{j\geq 1} \sqrt{j}|Z_{j-1} - Z_j|\,] < +\infty$;

c2) $n \sum_{j\geq n} (Z_{j-1} - Z_j)^2 \xrightarrow{a.s.} V$ for some random variable V.

In the following we verify the above conditions.

Condition c1). We observe that equality (3) can be rewritten as

$$Z_{j-1} - Z_j = \frac{R_j N_j (Z_{j-1} - X'_j)}{S_j}\,, \tag{9}$$

so that we find

$$|Z_{j-1} - Z_j| \leq \frac{k^2}{j}\,. \tag{10}$$

Therefore condition c1) is obviously verified.

Condition c2). We want to apply Lemma A.2 with $Y_j = j^2 (Z_{j-1} - Z_j)^2$. By the assumptions and inequality (10), we have $\sum_j j^{-2} E[Y_j^2] < +\infty$. Moreover, by equality (9), we have

$$E[Y_j|\mathcal{F}_{j-1}] = j^2 E[(Z_{j-1} - Z_j)^2|\mathcal{F}_{j-1}] = j^2 E[S_j^{-2} R_j^2 N_j^2 (Z_{j-1} - X'_j)^2 |\mathcal{F}_{j-1}],$$

and so (by iii)) we get the two inequalities

$$E[Y_j|\mathcal{F}_{j-1}] \geq \frac{j^2}{(S_{j-1} + k^2)^2} E[R_j^2] E[N_j^2 (Z_{j-1} - X'_j)^2 |\mathcal{F}_{j-1}]$$

$$E[Y_j|\mathcal{F}_{j-1}] \leq \frac{j^2}{S_{j-1}^2} E[R_j^2] E[N_j^2 (Z_{j-1} - X'_j)^2 |\mathcal{F}_{j-1}].$$

Since $S_n/n \xrightarrow{a.s.} Nm$ and $E[R_j^2]$ converges to q, it is enough to prove the almost sure convergence of $E[N_j^2 (Z_{j-1} - X'_j)^2 |\mathcal{F}_{j-1}]$ to $NZ(1-Z)$. To this purpose, we observe that we can write

$$E[N_j^2 (Z_{j-1} - X'_j)^2 |\mathcal{F}_{j-1}] = E\left[N_j^2 E[(Z_{j-1} - X'_j)^2 |\mathcal{G}_{j-1}]\,|\,\mathcal{F}_{j-1}\right]$$

and, by ii) and relation (8), the conditional expectation $E[(Z_{j-1}-X'_j)^2|\mathcal{G}_{j-1}]$ coincides with

$$
\begin{aligned}
&Z_{j-1}^2 + N_j^{-2}E[X_j^2|\mathcal{G}_{j-1}] - 2Z_{j-1}E[X'_j|\mathcal{G}_{j-1}] = \\
&Z_{j-1}^2 + N_j^{-2}\left[Z_{j-1}(1-Z_{j-1})(S_{j-1}-1)^{-1}N_j(S_{j-1}-N_j) + Z_{j-1}^2N_j^2\right] - 2Z_{j-1}^2 = \\
&Z_{j-1}(1-Z_{j-1})(S_{j-1}-1)^{-1}N_j^{-1}\left(S_{j-1}-N_j\right).
\end{aligned}
$$

Therefore we obtain

$$E[N_j^2(Z_{j-1}-X'_j)^2|\mathcal{F}_{j-1}] = Z_{j-1}(1-Z_{j-1})(S_{j-1}-1)^{-1}\left(S_{j-1}E[N_j|\mathcal{F}_{j-1}] - E[N_j^2|\mathcal{F}_{j-1}]\right),$$

which converges to $NZ(1-Z)$ (since $E[N_j^2|\mathcal{F}_{j-1}]$ is bounded by k^2 and $S_{j-1} \xrightarrow{a.s.} +\infty$). Hence $E[Y_j|\mathcal{F}_{j-1}]$ converges almost surely to V and, by Lemma A.2, condition c2) is satisfied.

The proof is so concluded.

Proof of Theorem 2. Thanks to what we have already proven in the previous proof, it suffices to verify that the following condition is satisfied (see Theorem A.3 applied to $Y_n = X'_n$):

c3) $n^{-1}\sum_{j=1}^n \left[X'_j - Z_{j-1} + j(Z_{j-1}-Z_j)\right]^2 \xrightarrow{P} U$ for some random variable U.

To this purpose, we apply Lemma A.2 with

$$Y_j = \left[X'_j - Z_{j-1} + j(Z_{j-1}-Z_j)\right]^2.$$

Indeed, by the assumptions and inequality (10), we have $\sum_j j^{-2}E[Y_j^2] < +\infty$. Moreover, from what we have already seen in the previous proof, we can get

$$j^2E[(Z_{j-1}-Z_j)^2|\mathcal{F}_{j-1}] \xrightarrow{a.s.} V,$$

$$E[(X'_j - Z_{j-1})^2|\mathcal{F}_{j-1}] \xrightarrow{a.s.} \eta Z(1-Z)$$

and, with a similar arguments,

$$
\begin{aligned}
2jE[(X'_j - Z_{j-1})(Z_{j-1}-Z_j)|\mathcal{F}_{j-1}] &= -2jE\left[S_j^{-1}R_jN_j(Z_{j-1}-X'_j)^2|\mathcal{F}_{j-1}\right] \\
&\xrightarrow{a.s.} -2N^{-1}Z(1-Z).
\end{aligned}
$$

Proof of Corollary 3. Assertion a) is proven in Corollary 3.4. in [3]. Let us prove assertion b) arguing as in [43].

Let A be a $\bigvee_n \mathcal{F}_n$-measurable event and set $I_n = E[I_A|\mathcal{F}_n]$. Then $I_n \xrightarrow{a.s.} I_A$. By Lemma A.1, we find

$$E\left[(I_A - I_n)\exp(it\sqrt{n}(Z_n - Z))|\mathcal{F}_n\right] \xrightarrow{a.s.} 0. \tag{11}$$

On the other hand, by Theorem 1, we have

$$E\left[I_n\exp(it\sqrt{n}(Z_n-Z))|\mathcal{F}_n\right] = I_nE\left[\exp(it\sqrt{n}(Z_n-Z))|\mathcal{F}_n\right] \xrightarrow{a.s.} I_A\exp(-(t^2V)/2). \tag{12}$$

Hence, from (11) and (12), we get

$$E\left[I_A\exp(it\sqrt{n}(Z_n - Z))|\mathcal{F}_n\right] \xrightarrow{a.s.} \exp(-(t^2V)/2)I_A. \tag{13}$$

In order to conclude, it is enough to fix $z \in (0,1)$, take $A = \{Z = z\}$ and observe that (13) implies almost surely

$$
\begin{aligned}
I_A \exp(-(t^2 V)/2) &= \lim_n E\big[I_A \exp(it\sqrt{n}(Z_n - Z))|\mathcal{F}_n\big] \\
&= \lim_n E\big[I_A \exp(it\sqrt{n}(Z_n - z))|\mathcal{F}_n\big] \\
&= \lim_n I_n \exp(it\sqrt{n}(Z_n - z)) = \lim_n I_A \exp(it\sqrt{n}(Z_n - z))
\end{aligned}
$$

and so almost surely

$$
I_A = |\lim_n I_A \exp(it\sqrt{n}(Z_n - z))| = I_A \exp(-(t^2 V)/2).
$$

Since we have $V > 0$ on A, it results $\exp(-(t^2V)/2) < 1$ on A for $t \neq 0$ and so we necessarily conclude that $P(A)$ is zero.

6. Statistical tools

6.1. **Asymptotic confidence intervals for the limit proportion.** By means of Theorem 1 and Theorem 2, we can construct asymptotic confidence intervals for the limit proportion Z. More precisely, under the assumptions of Theorem 1, also assume $k \leq a + b$ (so that $N_n \leq S_{n-1}$ for each n). If we are in the particular case when:

- for each n, the random variable N_n is independent of $\mathcal{F}_{n-1}$ and all the random variables N_n are identically distributed with mean value μ (so that $N = E[N_n] = \mu$ and $\eta = E[N_n^{-1}]$) and
- all the random variables R_n (that are independent by assumption iii)) are also identically distributed (so that $m = E[R_n]$ and $q = E[R_n^2]$),

then two asymptotic confidence intervals for Z are

$$
Z_n \pm q_{1-\frac{\alpha}{2}} \sqrt{\frac{V_n}{n}} \qquad M_n \pm q_{1-\frac{\alpha}{2}} \sqrt{\frac{W_n}{n}} \tag{14}
$$

where $q_{1-\frac{\alpha}{2}}$ is the quantile of order $1 - \frac{\alpha}{2}$ of the standard normal distribution and

$$
V_n = \frac{q_n}{m_n^2 \mu_n} Z_n(1 - Z_n), \qquad W_n = \left(\frac{2q_n}{m_n^2 \mu_n} + \eta_n - \frac{2}{\mu_n} \right) M_n(1 - M_n) \tag{15}
$$

with

$$
\begin{aligned}
m_n &= \frac{\sum_{j=1}^n R_j}{n}, \qquad & q_n &= \frac{\sum_{j=1}^n R_j^2}{n} \\
\mu_n &= \frac{\sum_{j=1}^n N_j}{n}, \qquad & \eta_n &= \frac{\sum_{j=1}^n N_j^{-1}}{n}.
\end{aligned} \tag{16}
$$

Note that the second interval does not depend on the initial composition of the urn, which could be unknown.

6.2. **The case of more urns.** Let $\mathcal{U}$ be a finite set. Every index $u \in \mathcal{U}$ labels an urn initially containing $a(u)$ balls of color A and $b(u)$ balls of color B. Each of the urn follows the dynamics described in the Section 2. For instance, according to the interpretations given in Section 1, we can see $\mathcal{U}$ as a set of different markets or different populations.

More precisely, we take a probability space $(\Omega, \mathcal{A}, P)$ and, on it, some random vectors $X_n = [X_n(u)]_{u\in\mathcal{U}}$, $N_n = [N_n(u)]_{u\in\mathcal{U}}$, $R_n = [R_n(u)]_{u\in\mathcal{U}}$ such that, for each $n \geq 1$, we have:

i) The conditional distribution of the random vector N_n given
$$[N_1, X_1, R_1, \ldots, N_{n-1}, X_{n-1}, R_{n-1}]$$
is concentrated on $\prod_{u\in\mathcal{U}}\{1, \ldots, S_{n-1}(u)\}$ where
$$S_{n-1}(u) = a(u) + b(u) + \sum_{j=1}^{n-1} N_j(u)R_j(u). \tag{17}$$

ii) The conditional distribution of the random vector X_n given
$$[N_1, X_1, R_1, \ldots, N_{n-1}, X_{n-1}, R_{n-1}, N_n]$$
is the product
$$\bigotimes_{u\in\mathcal{U}} \text{Hypergeom}(N_n(u), S_{n-1}(u), H_{n-1}(u)),$$
where $\text{Hypergeom}(N_n(u), S_{n-1}(u), H_{n-1}(u))$ denotes the hypergeometric distribution with parameters $N_n(u)$, $S_{n-1}(u)$ and $H_{n-1}(u)$ with
$$H_{n-1}(u) = a(u) + \sum_{j=1}^{n-1} X_j(u)R_j(u). \tag{18}$$

iii) The random vector R_n takes values in $(\mathbb{N}\setminus\{0\})^{card(\mathcal{U})}$ and it is independent of
$$[N_1, X_1, R_1, \ldots, N_{n-1}, X_{n-1}, R_{n-1}, N_n, X_n].$$

We set $Z_n(u)$ equal to the proportion of balls of color A in the urn u (immediately after the updating of the urn at time n and immediately before the $(n+1)$-th extraction), that is $Z_0(u) = a(u)/(a(u) + b(u))$ and
$$Z_n(u) = \frac{H_n(u)}{S_n(u)} \quad \text{for } n \geq 1.$$
Moreover we set
$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma\big(N_1, X_1, R_1, \ldots, N_n, X_n, R_n\big) \quad \text{for } n \geq 1\,,$$
and
$$\mathcal{G}_n = \mathcal{F}_n \vee \sigma(N_{n+1}) \quad \text{for } n \geq 0.$$

From condition ii) follows that $X_n(u)$ and $X_n(v)$ are $\mathcal{G}_{n-1}$-conditionally independent for $u \neq v$ and so, setting $X'_n(u) = X_n(u)/N_n(u)$ for each n and u, we have
$$\begin{aligned}
&E\left[(Z_{n-1}(u) - X'_n(u))\,(Z_{n-1}(v) - X'_n(v))\,|\mathcal{G}_{n-1}\right] = \\
&E[Z_{n-1}(u) - X'_n(u)|\mathcal{G}_{n-1}]E[Z_{n-1}(v) - X'_n(v)|\mathcal{G}_{n-1}] \\
&= 0.
\end{aligned} \tag{19}$$

It is worthwhile to note that, for a given n, we are not assuming the random variables $N_n(u)$ (resp. $R_n(u)$), with $u \in \mathcal{U}$, to be independent. For example, we can assume
$$N_n(u) = h(u) + F'_n \qquad R_n(u) = r(u) + F''_n$$

where $h(u)$, $r(u)$ are specific constants for each urn u and F'_n, F''_n are random factors that are common to all the urns.

Suppose now that there exists $k \in \mathbb{N} \setminus \{0\}$ with $N_n(u) \vee R_n(u) \leq k \leq a(u) + b(u)$ for each n and u and that the additional assumptions stated in Section 6.1 are satisfied for each u. Set $m(u) = E[R_n(u)]$, $q(u) = E[R_n(u)^2]$, $\mu(u) = E[N_n(u)]$ and $\eta(u) = E[N_n(u)^{-1}]$. Denoting by $M_n = [M_n(u)]_{u \in \mathcal{U}}$ the vector containing the empirical mean of $X'_j(u)$ up to time n for each urn u and by $Z = [Z(u)]_{u \in \mathcal{U}}$ the vector containing the almost sure limit of $Z_n(u)$ (and $M_n(u)$) for each u, we have as a consequence of (19) that, for any vector $\alpha = [\alpha(u)]_{u \in \mathcal{U}}$ of real numbers, the sequence $\sqrt{n}\langle \alpha, (Z_n - Z) \rangle$[3] converges in the sense of the almost sure conditional convergence with respect to $\mathcal{F}$ to $\mathcal{N}(0, \sum_{u \in \mathcal{U}} \alpha(u)^2 V(u))$, where $V(u) = \frac{q(u)}{m(u)^2 \mu(u)} Z(u)(1 - Z(u))$ and $\sqrt{n}\langle \alpha, (M_n - Z) \rangle$ converges stably to $\mathcal{N}(0, \sum_{u \in \mathcal{U}} \alpha(u)^2 (U(u) + V(u)))$, where $U(u) = \left(\frac{q(u)}{m(u)^2 \mu(u)} + \eta(u) - \frac{2}{\mu(u)} \right) Z(u)(1 - Z(u))$. Similarly as done in the previous section, these convergence results can be useful in order to get asymptotic confidence intervals for the linear combination $\langle \alpha, Z \rangle$ of the limit proportions $Z(u)$.

Finally, the above results can be employed in order to obtain asymptotic critical regions for tests. For instance, in order to perform a statistical test with

$$H_0 : \ m(u) \geq \mathrm{card}(\mathcal{U}')^{-1} \sum_{v \in \mathcal{U}'} m(v) \quad \text{against} \quad H_1 : \ m(u) < \mathrm{card}(\mathcal{U}')^{-1} \sum_{v \in \mathcal{U}'} m(v)$$

where $\mathcal{U}' \subset \mathcal{U}$ and $u \notin \mathcal{U}'$, we can use the asymptotic critical region

$$\left\{ \frac{\sqrt{\mathrm{card}(\mathcal{U}')^{-1} \sum_{v \in \mathcal{U}'} m_n(v)}}{\sqrt{m_n(u)}} \frac{\sqrt{n}\, |M_n(u) - Z_n(u)|}{\sqrt{U_n(u)}} > q_{1 - \frac{\alpha}{2}} \right\}$$

where

$$U_n(u) = \left(\frac{q_n(u)}{m_n(u)^2 \mu_n(u)} + \eta_n(u) - \frac{2}{\mu_n(u)} \right) Z_n(u)(1 - Z_n(u)) \tag{20}$$

with

$$\begin{aligned} m_n(u) &= \frac{\sum_{j=1}^n R_j(u)}{n}, \qquad & q_n(u) &= \frac{\sum_{j=1}^n R_j(u)^2}{n} \\ \mu_n(u) &= \frac{\sum_{j=1}^n N_j(u)}{n}, \qquad & \eta_n(u) &= \frac{\sum_{j=1}^n N_j(u)^{-1}}{n}. \end{aligned} \tag{21}$$

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[3] The symbol $\langle \cdot, \cdot \rangle$ denotes the scalar product between two vectors.

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Appendix A. Some auxiliary results

For reader's convenience, we state here some results used in the proofs.

Lemma A.1. (Th. 2 in [13] or a special case of Lemma A.2 in [21])
Let $\mathcal{F}$ be a filtration and set $\mathcal{F}_\infty = \bigvee_n \mathcal{F}_n$. Then, for each sequence (Y_n) of integrable complex random variables, which is dominated in L^1 and which converges almost surely to a complex random variable Y, the conditional expectation $E[Y_n|\mathcal{F}_n]$ converges almost surely to the conditional expectation $E[Y|\mathcal{F}_\infty]$.

Lemma A.2. (Lemma 2 in [9])
Let (Y_n) be a sequence of real random variables, adapted to a filtration $\mathcal{F}$. If $\sum_{j\geq 1} j^{-2}E[Y_j^2] < +\infty$ and $E[Y_j|\mathcal{F}_{j-1}] \stackrel{a.s.}{\longrightarrow} Y$ for some random variable Y, then

$$n\sum_{j\geq n}\frac{Y_j}{j^2} \stackrel{a.s.}{\longrightarrow} Y, \qquad \frac{1}{n}\sum_{j=1}^n Y_j \stackrel{a.s.}{\longrightarrow} Y.$$

Theorem A.3. (Special case of Th. 1 together with Prop. 1 in [9] and Th. 10 in [8])
Let (Y_n) be a bounded sequence of real random variables, adapted to a filtration $\mathcal{G} = (\mathcal{G}_n)$. Set

$$M_n = \frac{1}{n}\sum_{j=1}^n Y_j \quad \text{and} \quad Z_n = E[Y_{n+1}|\mathcal{G}_n].$$

Suppose that (Z_n) is a $\mathcal{G}$-martingale.

Then, $Z_n \stackrel{a.s./L^1}{\longrightarrow} Z$ and $M_n \stackrel{a.s./L^1}{\longrightarrow} Z$ for some real random variable Z. Moreover, $\sqrt{n}(Z_n - Z)$ converges in the sense of the almost sure conditional convergence with respect to $\mathcal{G}$ toward the Gaussian kernel $\mathcal{N}(0, V)$ for some random variable V, provided

c1) $E\left[\sup_{j\geq 1} \sqrt{j}\,|Z_{j-1} - Z_j|\,\right] < +\infty$,

c2) $n \sum_{j\geq n}(Z_{j-1} - Z_j)^2 \stackrel{a.s.}{\longrightarrow} V$.

If condition

c3) $n^{-1} \sum_{j=1}^{n} \left[Y_j - Z_{j-1} + j(Z_{j-1} - Z_j)\right]^2 \stackrel{P}{\longrightarrow} U$

is also satisfied for some random variable U, then

$$\left[\sqrt{n}\,\big(M_n - Z_n\big), \sqrt{n}(Z_n - Z)\right] \stackrel{stably}{\longrightarrow} \mathcal{N}\big(0,\, U\big) \otimes \mathcal{N}(0, V).$$

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